Force localization in contracting cell layers
Supplemental Material

Carina M. Edwards and Ulrich S. Schwarz

July 30, 2011

In the following, equations labelled by numbers refer to the corresponding equations in the Letter, while equations labelled with letters refer to equations contained in this Supplemental Material.

A Layer stress for contracting stripe

We first present an explicit formula for the internal stress in the contracting stripe case. In this case the displacement u is found in the manuscript to be

$$\frac{u}{l_0} = -P_0 \frac{\sinh(\gamma x/l_0)}{\gamma \cosh\gamma}.$$

Substituting this expression for u into the constitutive relation in one-dimension

$$F = \frac{hE_{\mathrm{c}}}{1-\nu^2}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{hE_{\mathrm{c}}}{2(1-\nu)}P \tag{A.1}$$

gives the internal stress $F(x)$ in the layer as

$$F = \frac{hE_{\mathrm{c}}P_0}{1-\nu^2}\left(1 - \frac{\cosh(\gamma x/l_0)}{\cosh\gamma}\right). \tag{A.2}$$

Eq. (A.2) is plotted in Fig. 2(b) in the manuscript for various values of γ.

B Uniformly contracting disc

The solution for the displacement u for a uniformly contracting disc is found in the manuscript to be

$$\frac{u}{r_0} = -\frac{1}{2\gamma}P_0(1+\nu)A(\gamma)I_1(\gamma r/r_0),$$
$$A(\gamma) = \left(I_0(\gamma) + \frac{\nu-1}{\gamma}I_1(\gamma)\right)^{-1},$$

which is plotted in Fig. SI(a) for different values of gamma. Note the qualitative similarity of this solution to that of the contracting stripe presented in Fig. 2.

In order to show the dependence of the solution on γ, we focus on the average displacement over the array, i.e.

$$\bar{u} \quad = \quad \frac{1}{\pi r_0^2}\int_D u\,\mathrm{d}S, \tag{B.1}$$

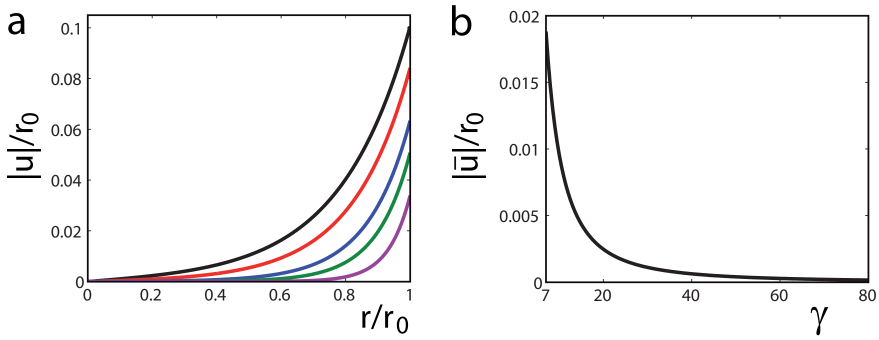

Fig. SI (a) Plot of radial displacement $|u|$ with γ =5, 6, 8, 10, 15 (top to bottom). (b) Plot of average displacement $|\bar{u}|$ scaled by cell sheet radius r_0 against localization parameter γ. In both plots $P_0 = 0.7$ and $\nu = 0.45$.

which is an often reported quantity in the literature. In Fig. SI(b) we plot this spatially averaged displacement as a function of γ. We conclude that average displacement sharply decreases with increasing stiffness, which suggests that to investigate the effect of substrate stiffness, it is advisable to examine parameter values at the lower end of the spectrum.

To calculate the internal stresses in the layer, we note that the only non-zero stresses are the radial F_{rr} and hoop $F_{\theta\theta}$ stresses, which are given by

$$F_{rr} = \frac{hE_c}{1+\nu}\left(\frac{1}{1-\nu}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{\nu}{1-\nu}\frac{u}{r}\right) - \frac{hE_c P}{2(1-\nu)}, \tag{B.2}$$

$$F_{\theta\theta} = \frac{hE_c}{1+\nu}\left(\frac{\nu}{1-\nu}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{1-\nu}\frac{u}{r}\right) - \frac{hE_c P}{2(1-\nu)}. \tag{B.3}$$

Thus to obtain expressions for the cellular stress we substitute the expression for u given above into Eqs. (B.2) and (B.2) with $P = -P_0$ to find that

$$F_{rr} = \frac{hE_c P_0}{2(1-\nu)}\left(1 - A(\gamma)\left(I_0\left(\frac{\gamma r}{r_0}\right) - \frac{(1-\nu)r_0}{\gamma r} I_1\left(\frac{\gamma r}{r_0}\right)\right)\right), \tag{B.4}$$

$$F_{\theta\theta} = \frac{hE_c P_0}{2(1-\nu)}\left(1 - A(\gamma)\left(\nu I_0\left(\frac{\gamma r}{r_0}\right) + \frac{(1-\nu)r_0}{\gamma r} I_1\left(\frac{\gamma r}{r_0}\right)\right)\right), \tag{B.5}$$

where A is also given above. At $r = r_0$, $F_{rr} = 0$ as required, but note that the hoop stress does not vanish at the boundary.

C Layers with more contractile rims

Adressing the case of non-uniform contractility, we consider in the manuscript a contraction P of the form

$$P = \begin{cases} -P_0 & x < x_1 \text{ or } r \leq r_1 \\ -P_1 & x_1 < x < l_0 \text{ or } r_1 < r < r_0 \end{cases},$$

with $P_1 > P_0$ so that the edge of the layer is more contractile than the central region. With this form of contraction it is possible to obtain analytical expressions for the resulting displacements u as noted in the main text. Here we present the exact solutions.

Solving Eq. (4) with zero-stress at the outer boundary, $u(0) = 0$ and continuity of u and F across $x = x_1$ we find that

$$\frac{u}{l_0} = \begin{cases} -\frac{1}{2\gamma\cosh\gamma} P_1(1+\nu)\left(\sinh(\gamma x/l_0) - (1 - P_0/P_1)\cosh\gamma\,(1 - x_1/l_0)\sinh(\gamma x/l_0)\right) & x < x_1, \\ -\frac{1}{2\gamma\cosh\gamma} P_1(1+\nu)\left(\sinh(\gamma x/l_0) - (1 - P_0/P_1)\cosh\gamma\,(1 - x/l_0)\sinh(\gamma x_1/l_0)\right) & x_1 < x < l_0. \end{cases} \tag{C.1}$$

Equivalently solving Eq. (7) with zero-stress at the outer boundary, $u(0) = 0$ and continuity of u and F across $r = r_1$ we find that the solution is

$$\frac{u}{r_0} = \begin{cases} -\frac{1}{2\gamma} P_1 (1+\nu) a I_1(\gamma r/r_0) & r < r_1, \\ -\frac{1}{2\gamma} P_1 (1+\nu) \left(b I_1(\gamma r/r_0) - c K_1(\gamma r/r_0) \right) & r_1 < r < r_0, \end{cases} \tag{C.2}$$

where I_1 and K_1 are the modified Bessel functions of first and second kind, respectively. The constants a, b, c are given by

$$a = \frac{1}{A(\gamma)} - \frac{1 - P_0/P_1}{A(\gamma r_1/R)} + \frac{I_1(\gamma r_1/R)\,(1 - P_0/P_1)}{K_1(\gamma r_1/R) A(\gamma r_1/R) + I_1(\gamma r_1/R) B(\gamma r_1/R)} \left(\frac{B(\gamma r_1/R)}{A(\gamma r_1/R)} - \frac{B(\gamma)}{A(\gamma)} \right), \tag{C.3}$$

$$b = \frac{1}{A(\gamma)} \left(1 - \frac{I_1(\gamma r_1/R) B(\gamma)\,(1 - P_0/P_1)}{K_1(\gamma r_1/R) A(\gamma r_1/R) + I_1(\gamma r_1/R) B(\gamma r_1/R)} \right), \tag{C.4}$$

$$c = \frac{I_1(\gamma r_1/R)\,(1 - P_0/P_1)}{K_1(\gamma r_1/R) A(\gamma r_1/R) + I_1(\gamma r_1/R) B(\gamma r_1/R)}, \tag{C.5}$$

where

$$A(x) = I_0(x) + \frac{(\nu - 1)}{x} I_1(x), \tag{C.6}$$

$$B(x) = K_0(x) - \frac{(\nu - 1)}{x} K_1(x). \tag{C.7}$$

The solution Eq. (C.2) is plotted in the Letter in Fig. 3.

Both solutions (C.1) and (C.2) show the same qualitative behaviour as described in the main text. Both exhibit a kink at the interface between the contractile regions, which becomes more prominent as P_0/P_1 decreases. As the outer rim begins to dominate the behaviour of the system, it is also becomes possible that there will be positive displacements within the layer, although the outer rim will always contract. Due to the greater tractability of the solution (C.1) it is possibility in this case to determine when this will occur analytically for this case and we find that positive displacements are possible if

$$P_0 \quad < \quad P_1 \left(1 - \frac{1}{\cosh \gamma\,(1 - x_1/x_0)} \right). \tag{C.8}$$

Note that as the extent of the contractile rim decreases its ability to drag the inner region to the right also decreases, indeed for $x_1/x_0 = 1 - \epsilon$, with $\epsilon \ll 1$, the critical value is $P_0 \sim \epsilon^2 \gamma^2 P_1/2$. A similar dependence on the extent of the more mechanically active region is also observed for the contractile disc, as can be seen in Fig. 3(b) in the main text.