Lecture notes on Quantum Field Theory

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Contents

*This is a draft version of the lecture notes. If you notice typos, points that remain unclear, mistakes or similar, please send an email to f.homburg@thphys.uni-heidelberg.de and a.eichhorn@thphys.uni-heidelberg.de.

0 Preliminaries

Literature

These lecture notes combine material from several sources. Some topics in these lecture notes are treated in all books listed below; for some, a particular book or selection of books is better suited. In this case, this will be indicated in the corresponding chapter.
In particular, you might find it useful to also read up on the topics in the following books as well as lecture notes (available online):

- Srednicki, Quantum Field Theory
- Ryder, Quantum Field Theory
- Gelis, Quantum Field Theory
- Schwartz, Quantum Field Theory and the Standard Model
- Peskin/Schröder, Quantum Field Theory
- Nastase, Quantum Field Theory
- Fradkin, Quantum Field Theory
- There are many other books on QFT and it is often a matter of personal taste, which one is most useful.
- Lecture notes on QFT by D. Tong (Cambridge University), A. Hebecker (Heidelberg University), T. Weigand (from the QFT courses at Heidelberg University)

Many topics are treated to the greatest level of depth in the QFT books by Weinberg. However, for a first encounter with a topic, the books are usually not useful, but rather become helpful later on, when one has already learned about a topic and wants to come back to it to learn more about it.
There is also the book "Quantum field theory in a nutshell" by Zee, which focuses more on some conceptual aspects rather than technical points and it can be a useful addition to the above list of literature.

Mini-exercises

The best way to learn quantum field theory is to do calculations yourself, and think and discuss about concepts yourself. Therefore, each lecture has at least one "mini-exercise", which you will work on during the lecture. This gives you the opportunity to engage more actively with the material and notice when you have questions. You will likely not always have time to finish the mini-exercise during class. Therefore, solutions will not only be provided on the blackboard, but are also available in the back of the lecture notes. They will be made available in the update of the lectures notes that will be made online after each lecture.

1 Introduction

1.1 Motivation: Why quantum field theory?

Quantum mechanics is a non-relativistic theory. This results in a question, namely:

→ What happens to systems in which quantum effects and relativistic effects are important?
There is a heuristic argument that points us towards how relativistic quantum physics differs from quantum mechanics. From the standard Heisenberg uncertainty principle, one can motivate an uncertainty relation between energy E and time t, namely $\Delta E \Delta t \geqslant \frac{\hbar}{2}$. In a relativistic setting, we can combine this with $E = mc^2$, which we know from special relativity.
$\Longrightarrow$ We expect that particle number is never fixed in a system, because, for short enough time durations, energy is not constant, but fluctuates and these fluctuations in energy translate into fluctuations in particle number. We call these fluctuations "virtual" particles.
$\Longrightarrow$ We cannot work with a wavefunction for a fixed number of particles, as we did in quantum mechanics. Instead, we need a formalism in which the particle number can change in a system over time, and in which the presence of virtual particles is accounted for.

We can also see the incompatibility between special relativity and quantum mechanics in a different way:

→ Special relativity requires that two measurements that are done at spacelike separation, must be independent in order not to violate causality. In Quantum Mechanics, independence of measurements is encoded in commuting operators. However, the notion that spacelike separated operators commute is not naturally built into QM.

$\Longrightarrow$ We need to adapt our formalism.

How should the new formalism look like?
To go beyond wavefunctions for fixed numbers of particles, we need a (mathematical) quantity that is more fundamental than particles, i.e., particles should be a *derived* notion.
We take inspiration from electrodynamics, because electrodynamics can be formulated in a relativistic way. At the same time, we know from the photo-electric effect, that there are particles in electrodynamics, namely photons. Thus, it is a useful guide to point us to the type of formalism that we should develop. Electrodynamics is a field theory, i.e., the fundamental quantity is a *field*, i.e., a quantity that takes on values at each spacetime point.
From experiments, we already know that photons (the corresponding particles) are derived from the field, in fact, they correspond to (quantized) excitations of the field. This can, e.g., be seen in laser experiments, in which the power incident on a screen is recorded. As the intensity of the laser is lowered, the power arrives in discrete, "quantized packages", the photons.
In order to be compatible with special relativity, we need to build a theory which has Lorentz invariance built into it, just like the relativistic formulation of electrodynamics has.

What will this type of theory be able to describe?

- elementary particles and their interactions, in particular the Standard Model of particle physics.

- any setting in which particle number is not conserved, e.g., condensed-matter-systems in which we are interested in effective (not fundamental) excitations, such as, e.g., phonons, or Cooper-pairs in superconductivity.
- if the energy of the system is low enough, the formalism that we are developing is even sufficient to understand the quantum properties of gravity.

Note: in our current understanding of cosmology, the origin of all structures in the universe (galaxies, galaxy clusters ...) are quantum fluctuations of the fundamental fields in the early universe. Ultimately, we thus owe our existence to the physics of QFT!

1.2 Why learning quantum field theory is hard

Quantum field theory is not an easy subject. This has several reasons. First, the quantities that we are dealing with are often abstract and more difficult to develop an intuitive understanding of than, for instance, systems in classical mechanics. Second, we need to develop an entirely new formalism to describe quantum fields, in which we bring together classical field theory and quantum theory. In other words, we are learning a (mathematical) language in which to describe the systems that we are interested in, and, just like with any other new language, learning it can be hard and it takes some time until the concepts start to feel familiar and intuitive.
However, you should not feel discouraged by this or think about giving up. Rather, if you have questions and/or doubts, bring them up with the lecturer (either after the lecture, or by email to eichhorn@thphys.uni-heidelberg.de) or to your tutor, or to the head tutor, Zois Gyftopolous (gyftopolous@thphys.uni-heidelberg.de). The whole team of lecturer and tutors is here to support you in learning and understanding quantum field theory!

1.3 Why learning quantum field theory is absolutely worth it

Quantum field theory provides the framework for the most advanced and deepest understanding of fundamental physics that we have. Therefore, it is like a key with which we can unlock fascinating insights into elementary particles and their properties. Thus, some of the highlights that await us this term are:

- understanding how powerful symmetries are and how we can deduce properties of elementary particles from an understanding of the Lorentz group and how we can deduce the existence of the electromagnetic field from thinking about symmetries
- understanding were the Pauli principle for Spin-1/2-particles comes from
- understanding why antiparticles must exist in order for causality to not be violated
- understanding that the vacuum is not a boring state of "nothing", but is a highly non-trivial state which results in a force between conducting plates ("Casimir force") or the scattering of photons off each other (unlike in classical electrodynamics, in which the equations of motion for the gauge field are linear and electromagnetic waves do not interact)
- and much more!

1.4 Classical field theory

We have already emphasized the role and importance of *symmetries*, so we will spend some more time developing the *mathematics of symmetries*, namely groups and their representations. First, however, we need to establish some of the notions that form the basis of this course, namely fields and their classical description.

A field takes a value at each spacetime point. Examples that you may already know include

- the E- and B-field, $E(\vec{x},t), B(\vec{x},t)$, which are 3-vectors.
- the density in hydrodynamics, $\rho(\vec{x},t)$, which is just a one-component function.
- the gauge field $A_\mu(\vec{x},t)$ in electrodynamics, which is a 4-vector.

To describe their dynamics, we start from an action S, which is a *functional*, i.e., its argument is a function (and it maps to the real numbers).
For instance, in the relativistic way of writing electrodynamics, we have

$$\begin{aligned} S[A] &= \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}, \\ &= \int d^4x \mathcal{L}_{\mathrm{ED}}, \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{1}$$

We denote functionals with square brackets around their arguments, which are functions. $\mathcal{L}_{\mathrm{ED}}$ is the *Lagrange density*. It is not a functional, because it does not depend on the full function (in this case, the field at all spacetime points), but is just a function of the spacetime-coordinates, through its dependence on the field at a point.
To establish some of the key notions, we will use a scalar field, conventionally denoted by $\phi(\vec{x},t)$. An example for scalar fields relevant in nature is the Higgs field in the Standard Model; hypothetical scalar fields include the inflaton field (that drives the (conjectured) inflationary phase in the early universe), and proposals for dark matter (e.g., the axion, which is, to be more precise, a pseudoscalar). Scalars that can be collective degrees of freedom also play a role in many condensed-matter systems, starting from the Ising model.
The Lagrange density $\mathcal{L}$ depends on the field $\phi(\vec{x},t)$ and its derivatives, $\partial_\mu\phi(\vec{x},t)$, $\partial_\mu\partial_\nu\phi(\vec{x},t)$ ... and is a priori completely arbitrary. We will make two assumptions:

- the Lagrange density is *local*, i.e., it depends on fields and their derivatives at one point and it only depends on a finite number of derivatives. (We call this local, because a derivative always compares a field at a point to its (infinitesimally removed) neighboring point. An infinitely high power of derivatives thus involves fields a finite distance apart.)
  This has two motivations: First, observationally, local interactions seem to describe nature very well; e.g., in the LHC detectors, one can see that particles interact locally. Second, non-local interaction may get into conflict with causality, because non-localities may mean interactions at spacelike distances.

- We assume that the Lagrange density does not have higher than second derivatives in time. The reason is Ostrogradsky's theorem, which is a theorem in classical mechanics and states that, (under a non-degeneracy condition), a Hamiltonian that contains higher-than-second-order time derivatives is unbounded from below. This may- but need not!- make the the-

ory dynamically unstable. Because this theorem implicitly underlies the formulation of Lagrangians in many settings (classical mechanics, classical field theory, quantum field theory), we will take a closer look at it in the exercises.[1]

The Lagrangian

$$L = \int d^3x \mathcal{L}, \tag{2}$$

is the spatial integral of the Lagrange density. We will often work with $\mathcal{L}$, because it makes the equal treatment of space and time, that we want in a relativistic theory, manifest. It is often called "the Lagrangian" in a slight abuse of naming conventions.

$\mathcal{L}$ consists of two parts, a kinetic part, T, that depends on derivatives, and a potential, V,

$$\mathcal{L} = T - V. \tag{3}$$

We will often focus on

$$T = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \tag{4}$$

and

$$V = \frac{1}{2}m^2\phi^2 + \lambda\phi^4, \tag{5}$$

where in V we assumed that we can Taylor-expand $V(\phi)$ around a minimum ϕ_0 and we can set $\phi_0 = 0$ and $V(\phi_0) = 0$ without loss of generality. We further assume a symmetry $\phi \to -\phi$, so that there is no ϕ^3 present, which would render $V(\phi)$ unbounded from below. We call m the mass, because we will see that the equations of motion imply $p^2 = m^2$ for the square of the four-momentum, if the term $m^2\phi^2$ is present in the Lagrangian. The quartic term, $\lambda\phi^4$ leads to non-linear equations of motion, i.e., it describes *interactions* of the field (and the corresponding particles) with itself. The strength of these interactions is parameterized by the coupling λ. In the next few lectures, we focus on just the mass term.

Our choice of T requires a bit more justification: The kinetic part describes how the field changes in space and time, thus it must contain a derivative, and $\partial_\mu\phi$ is the building block to use. In order to have a Lorentz-invariant expression, we must contract the open index and the only other 4-vector we have is another derivative. Thus, up to rescalings of the term, we have a unique lowest order action in ϕ

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\,\eta^{\mu\nu} - \frac{m^2}{2}\phi^2\right), \tag{6}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ in our conventions, which most QFT books use. Many GR books use $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The overall sign is pure convention; the difference in signs between the time part and the spatial part is physics.

Mini-Exercise 1. We made the statement that we can set a constant and a linear term in $\mathcal{L}$ to zero without loss of generality. For the constant term, this is because the equations of motion follow from minimizing the action and the field value that minimizes S does not depend on whether or not there is a constant shift in S.

[1] You have probably encountered or will encounter many examples where the Lagrangian does not have higher than second order time derivatives. Electrodynamics is one example, General Relativity another, and classical mechanics is full of examples. Note however that there are subtleties and there are counterexamples to the intuition that a Hamiltonian that is unbounded from below leads to instabilities.

For the linear term, we can always remove it by a change of our field variable (which you can think of as analogous to a change in coordinates in class. mech.)
Show this! Start with

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - C\phi - \frac{1}{2}m^2\phi^2. \tag{7}$$

Define $\varphi = \phi + \gamma$. What is the choice of γ, such that

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + const\,? \tag{8}$$

Solution.

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - C(\varphi-\gamma) - \frac{1}{2}m^2(\varphi-\gamma)^2 \\ &= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \underline{C\varphi} + C\gamma - \frac{1}{2}m^2\varphi^2 + \underline{m^2\varphi\gamma} - \frac{1}{2}m^2\gamma^2\end{aligned}$$

Define $\gamma = \frac{C}{m^2}$:

$$\rightarrow \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{C^2}{m^2} - \frac{1}{2}m^2\varphi^2 - \frac{1}{2}m^2\frac{C^2}{m^2}.$$

The equations of motion follow from extremizing the action, i.e., we perform a variation of the action (i.e., a variation of the field, $\phi \rightarrow \phi+\delta\phi$, by some arbitrary amount $\delta\phi$). We set the variation of the action to zero, just like, when we are searching for the minimum of a function, we are setting its first derivative (analogous to the variation of the argument of the function) to zero:

$$\begin{aligned}0 = \delta S &= \delta\int d^4x\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right) \\ &= \int d^4x\big((\partial_\mu\phi)\eta^{\mu\nu}(\partial_\nu\delta\phi) - m^2\phi\delta\phi\big) \\ &= \int d^4x\big(-(\partial_\nu\partial_\mu\phi)\eta^{\mu\nu}\delta\phi - m^2\phi\delta\phi\big) \\ &= \int d^4x\big(-(\partial_\nu\partial_\mu\phi)\eta^{\mu\nu} - m^2\phi\big)\delta\phi,\end{aligned} \tag{9}$$

where in the second-to-last step we used partial integration and where we assume that $\delta\phi = 0$ at $x \rightarrow \pm\infty$. Because $\delta\phi$ is an arbitrary variation, to satisfy Eq. (9), the factor $-\partial_\nu\partial_\nu\phi\eta^{\mu\nu} - m^2\phi$ must be zero.
This is the Klein-Gordon equation,

$$\boxed{\partial^2\phi + m^2\phi = 0,} \tag{10}$$

with $\partial^2 = \partial_\mu\partial_\nu\eta^{\mu\nu}$. The Klein-Gordon equation is a relativistic, massive wave equation.
For the Lagrangian, $\delta S = 0$ translates into the Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}\right) = 0. \tag{11}$$

The solutions to the equations of motion are spanned by plane waves,

$$\phi(x) = \phi_0\cos(kx), \qquad \text{(assuming } \phi(x) = \phi(-x)\text{)} \tag{12}$$

with the shorthand $kx = k_\mu x^\mu$ and the relativistic, massive dispersion-relation $k_\mu k^\mu = k^2 = m^2$.

Later on, a starting point for one quantization scheme (path-integral quantization) will be the action, but the starting point for another quantization scheme (canonical quantization) will be the Hamiltonian.

Just as in classical mechanics, where we define $p = \frac{\partial L}{\partial \dot{q}}$, the canonically conjugate momentum, and $H(p,q) = p\dot{q} - L$, in quantum field theory we define $\pi(\vec{x})$, the canonically conjugate field. (Note: it is the canonically conjugate field to ϕ, but has nothing to do with the momentum of the particles that we will describe. It is sometimes called the (canonically conjugate) momentum field, because it arises in the generalization of the Hamiltonian formalism to QFT and it generalizes the momentum of a particle, which is the canonically conjugate variable to the position.) Its definition is

$$\pi(\vec{x}) = \frac{\delta L}{\delta \dot{\phi}(\vec{x})}, \tag{13}$$

which is a functional derivative, i.e., a derivative with respect to a function. Just like $\frac{\partial x}{\partial x} = 1$, we have

$$\frac{\delta \phi(\vec{x})}{\delta \phi(\vec{y})} = \delta^3(\vec{x} - \vec{y}). \tag{14}$$

Thus, for the Lagrangian in Eq. (6), we obtain

$$\begin{aligned} \pi(\vec{x}) &= \frac{\delta}{\delta \dot{\phi}(\vec{x})} \int d^3y \left(\underbrace{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2}_{\partial_\mu \phi \partial^\mu \phi} - \frac{1}{2} m^2 \phi^2 \right) \\ &= \int d^3y \Big(\dot{\phi}\, \delta^3(\vec{x} - \vec{y}) \Big) = \dot{\phi}(\vec{x}). \end{aligned} \tag{15}$$

Thus, to calculate the Hamiltonian, we can use that $\dot{\phi}$ can be substituted by π. We obtain the Hamiltonian of the system as

$$\begin{aligned} H &= \left(\int d^3x\, \pi \dot{\phi} \right) - L \\ &= \int d^3x \left(\pi^2 - \left(\frac{\pi^2}{2} - \frac{1}{2}\left(\vec{\nabla}\phi\right)^2 - \frac{m^2}{2}\phi^2 \right) \right) \\ &= \frac{1}{2} \int d^3x \left(\underbrace{\pi^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2}_{2\mathcal{H}} \right) \\ &= \int d^3x\, \mathcal{H}, \end{aligned} \tag{16}$$

where we defined the Hamiltonian density $\mathcal{H}$.

2 The importance and the mathematics of symmetries

Useful literature for this chapter is the following: There are books on group theory in physics and more specifically particle physics, e.g., "Group theory in physics" by Wu-Ki Tung and "Lie algebras in particle physics" by Howard Georgi.
QFT books also cover discussions of symmetry groups, for instance: Schwartz, chapter 2, covers the basics of Lorentz transformations, and group theory basics for the Lorentz group are discussed in 10.1. The Lorentz group and its Lie algebra generators are also discussed in Srednicki, chapter 2. Gelis (chapter 7.1) summarizes Lie groups and Lie algebras.
Symmetries are one of the most important foundational elements in QFT. This becomes obvious from many examples:

i) In particle physics, the various mesons and baryons are organized into sets, e.g., the eight lightest mesons are grouped into the meson octet according to the "eightfold way" which is based on a so-called "$SU(3)$ flavor symmetry". Historically, this type of organization into sets according to symmetries was central in predicting new particles.

ii) You might have heard that the Standard Model is an $SU(3) \times SU(2) \times U(1)$ gauge theory. Specifying this symmetry already fixes a large part of the Standard Model particle content and the allowed interactions between particles.

iii) In condensed matter, phase transitions are associated with spontaneous *breaking of symmetries*. For instance, in a ferromagnet, at high enough temperature, there is no macroscopic magnetization, which means that there is full rotational symmetry for each of the microscopic spin vectors. At low temperature, in the magnetized phase, rotational symmetry is broken, because the macroscopic magnetization spontaneously selects one spatial direction. More generally, by knowing the symmetries that the degrees of freedom in a condensed-matter system obey, we can already figure out which phases and phase transitions there could be.

iv) Lorentz symmetry (or its generalization, Poincaré symmetry, which adds translations (in space and in time)) determine much of the properties of elementary particles and their interactions and much of the mathematical structure of QFT.
For instance, the fact that we characterize elementary particles by their mass and their spin is a direct consequence (as we will work out) follows from considering the Poincaré group.

v) ...

This motivates us to dive into the *mathematics of symmetries*, because this appears to be the language in which large parts of nature can be described.

2.1 Symmetries are described by groups

It turns out that there exists a mathematical structure that is exactly adapted to formalizing symmetries, and that is a group.
Definition of a group:

A group G is a set of elements $G_i \in G$, together with a "multiplication" $\cdot$, such that

$$G_i \cdot G_j = G_k, \qquad G_k \in G \quad \forall\, G_i, G_j \in G. \tag{17}$$

This means that we can combine two elements of the group by the multiplication and we obtain another element of the group. The multiplication law satisfies

- associativity

$$(G_i \cdot G_j) \cdot G_k = G_i \cdot (G_j \cdot G_k) \quad \forall\, G_i, G_j, G_k \in G. \tag{18}$$

- $\exists$ identity element E, s.t.

$$G_i \cdot E \in G \qquad \forall\, G_i \text{ and } E \in G. \tag{19}$$

- inverse element

$$\forall\, G_i \in G\, \exists\, G_i^{-1} \in G, \text{ s.t. } G_i \cdot G_i^{-1} = E. \tag{20}$$

Note that the identity element is unique, as is the inverse for each element.
Let's parse this definition and the intuition behind the various requirements in physics language, using rotations as an example and thinking of a spherically symmetric system:

- Two rotations can be performed consecutively, yielding a third rotation (about a different axis). This is the multiplication law which allows us to combine group elements into new group elements.
- When three rotations are performed, either the 1st and 2nd or 2nd and 3rd can be combined, such that the consecutive execution of the three of them is equal in any of the two combinations. (Note that we must not reverse the *order* of the three rotations, because the group is not commutative.)
- There is an identity element, namely rotation by 0° (or no rotation).
- For each rotation, we can reverse the sense of rotation to rotate back, such that the combination of rotation and inverse rotation yields no rotation.

You may already know that rotations can be represented by matrices, such that, e.g., the identity is the unit matrix and the inverse element is the inverse matrix.

We will encounter two *mathematically distinct* sets of groups that encode symmetries in QFT:

1) discrete groups (with a finite set of elements), for instance reflections about a plane (has three elements: the reflection, its inverse, and the identity).

2) continuous groups, which are Lie groups. The rotation group is an example. It is continuous, because it has infinitely many group elements (rotations by different angles) and "neighboring" rotations only differ infinitesimally.

We will also encounter three *physically distinct* types of groups[2]

[2]There is a theorem, the Coleman-Mandula theorem, that says that, under some assumptions, there are no symmetry groups that mix spacetime symmetry transformations with internal symmetry transformations. The realization that, by violating the assumptions, one can get around this theorem, and is then required to introduce so-called "super-partners" led to the development of supersymmetry, which we will not treat in this course, but which is a very interesting mathematical developments worth understanding. In nature, supersymmetry is realized in some low-dimensional settings in condensed-matter theory, but does not appear to be realized in particle physics. It is, however, instrumental in one approach to quantum gravity, namely in string theory.

a) spacetime symmetry groups, which can either be continuous (like the Lorentz group, $SO(1,3)$), or discrete (like time-reversal symmetry which maps the time t to $-t$)

b) internal symmetry groups, where "internal" here means that the symmetry does not act on space and time (like, e.g. the Lorentz group), but only on the field. These come in two different versions:

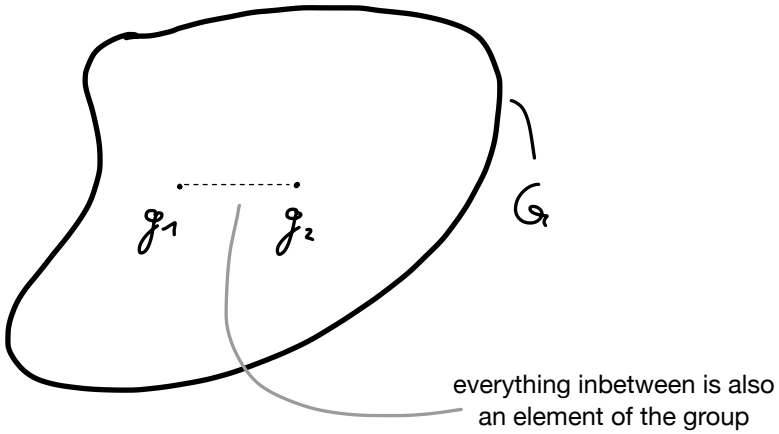
   i) global internal symmetries (like the $\mathbb{Z}_2$-symmetry $\phi \to -\phi$ that we imposed on scalar field theory to ensure that there is no ϕ^3 term in the scalar potential, or the $SO(3)$ symmetry that is imposed on the scalar field in the Heisenberg model that describes phase transitions in certain materials).
   Global means that the symmetry transformation is the same for the field at all spacetime points.

   ii) local internal symmetries (like the $U(1)$ gauge symmetry of electromagnetism).
   Local here means that the symmetry transformation can be different at different spacetime points (even if it doesn't act on the spacetime itself).

Some of these notions may seem a little abstract at the moment. They will become clearer as we develop our understanding of group theory and come up with examples.
The most relevant groups for us will be *Lie Groups*.

2.2 Lie groups

These are groups in which the group elements form not just a set, but a differentiable manifold (which is a collection of points such that each point has an open neighborhood that is equivalent to $\mathbb{R}^n$ and which can be covered by coordinate charts that overlap partially).
This means that the group is *continuous*, such that you can always find a group element infinitesimally close to any given element. Intuitively, we can see directly that the group of rotations should be such a continuous group, because we can always rotate by an arbitrarily small amount and thus find rotations which are only infinitesimally different from each other.



Examples:

- $U(1)$ is the group of all unitary 1×1 matrices, i.e.,

$$G = e^{i\alpha}, \qquad \alpha \in \mathbb{C}. \tag{21}$$

  The corresponding manifold is the circle (of radius 1) in the complex plane.
  As a global symmetry, the phase α of the transformation does not depend on the spacetime point. As a local symmetry, α is upgraded to a function $\alpha(x^\mu)$. We will explore the consequences of this soon. In fact, this group determines the properties of photons and their interactions with charged particles.

- $SU(2)$ is the group of 2×2 unitary matrices with determinant 1. The corresponding manifold is the 3-sphere, S^3.
  To see this, we write

$$U^\dagger U = \mathbb{1} \implies U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad \text{with } |a|^2 + |b|^2 = 1 \text{ for } a, b \in \mathbb{C} \tag{22}$$

  (Check:

$$U^\dagger = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \quad \text{and} \quad U^\dagger U = \begin{pmatrix} |a|^2 + |b|^2 & \cancel{a^* b - b a^*} \\ \cancel{b^* a - a b^*} & |a|^2 + |b|^2 \end{pmatrix} = \mathbb{1}) \tag{23}$$

  Now we write both complex numbers in terms of real and imaginary part,

$$a = x + iy, \qquad b = z + it \tag{24}$$

  $\implies |a|^2 + |b|^2 = |x|^2 + |y|^2 + |z|^2 + |t|^2 = 1$ parametrizes the group manifold $SU(2)$, where $x, y, z, t \in \mathbb{R}$.

This is the equation defining a unit 3-dimensional sphere embedded in 4-dimensional space, i.e., S^3.

SU(2) is the symmetry group determining the properties of the weak gauge bosons (W^+, W^- and Z) and their interactions with the fermions in the Standard Model.

With a Lie group comes a Lie algebra $\mathfrak{g}$, $\mathrm{Lie}(G) = \mathfrak{g}$. Knowing about the Lie algebra is useful, because all properties of the Lie group follow from knowing the so-called *generators* of the Lie algebra and their commutation relations.

A *Lie algebra* is a vector space $\mathfrak{g}$ with a bilinear, antisymmetric map:

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \qquad (a, b) \mapsto [a, b] = -[b, a] \tag{25}$$

(that we suggestively write in the same notation that we use for the commutator) that satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0. \tag{26}$$

We will only need matrix groups and matrix algebras. For matrix Lie groups, the relation between group and algebra is given through the *exponential map*: For $a_i \in \mathfrak{g}$,

$$G_i = \exp(a_i), \tag{27}$$

(defined through its Taylor series) is a group element. Each group element (in the so-called identity component of G) can be written in such a way. For $0 \in \mathfrak{g}$, $\mathbb{1} = \exp(0) \in G$.

We can find a basis in $\mathfrak{g}$ and these elements of the Lie algebra are called the *generators*. Having this basis of generators, we can construct every group element through the exponential map.

Example: Rotation group $SO(3)$

$SO(3)$ is the group of special orthogonal $3x3$ matrices, i.e., matrices which are orthogonal, so $\mathrm{Rot}\,\mathrm{Rot}^\top = \mathbb{1}$, where $\mathrm{Rot}^\top$ denotes the transposed matrix, and special, i.e., their determinant is $+1$. They describe rotations, because we can check that the requirement that a rotation leaves the length of a vector invariant requires $\mathrm{Rot}\,\mathrm{Rot}^\top = \mathbb{1}$. To check this, consider a spatial vector, with components x^i. Under a rotation, it is mapped to

$$x^i \to x^{i'} = \mathrm{Rot}^i{}_k\, x^k. \tag{28}$$

We require that its length stays invariant, so that

$$x^i\, x^j\, \delta_{ij} = x^{i'} x^{j'} \delta_{ij} = \mathrm{Rot}^i{}_k\, \mathrm{Rot}^j{}_l\, x^k\, x^l \delta_{ij}. \tag{29}$$

Thus, $\mathbb{1} = \mathrm{Rot}^i{}_k\, \mathrm{Rot}^j{}_l\, \delta_{ij} = \mathrm{Rot}^T\, \mathrm{Rot}$. This is in particular realized by matrices of the form

$$\mathrm{Rot}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \tag{30}$$

and analogously for rotations about the y and the z-axis.

Claim: $\mathrm{Lie}(G) = \{\text{antisymmetric } 3 \times 3 \text{ matrices}\}$

Mini-Exercise 2. Check that $\mathrm{Rot}\,\mathrm{Rot}^\top = \mathbb{1}$, as required for $\mathrm{Rot} \in SO(3)$, is realized by $\mathrm{Rot} = \exp(T)$, if $T_{ij} = -T_{ji}$.

Solution.

$$\mathrm{R}\,\mathrm{R}^\top = \exp(T)\exp(-T) = \mathbb{1},$$

because

$$\mathrm{R}^\top = (\exp(T))^\top = \exp T^\top = \exp(-T).$$

Let's see how we can reconstruct the group elements, i.e., the Rot matrices, from the Lie algebra generators. An antisymmetric 3×3 matrix with real components (so that Rot is real), has 3 independent components, so we have three basis elements

$$T_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{31}$$

and analogously for the other two *generators* of the Lie algebra. Now we can write a rotation about the x-axis as

$$\mathrm{Rot}_x = \exp(\theta\, T_x) = \mathbb{1} + \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \mathcal{O}(\theta^2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta \\ 0 & \theta & 1 \end{pmatrix} + \mathcal{O}(\theta^2), \tag{32}$$

which is clearly the infinitesimal version of the rotation matrix given above.

What will be crucial in our construction of QFT is the notion of *representations* of groups and algebras. For instance, we will construct the spin-0, spin-1/2 and spin-1 representations of the Lorentz group to describe the Higgs field, the electron and the photon in the Standard Model, or various excitations in condensed-matter systems.
Intuitively, a representation is a set of objects which satisfy the same multiplication rules as the abstract group elements, i.e., they are often matrices, for which the multiplication satisfies the combination rules that the group elements satisfy.
More formally, a representation R of a group is a map $G \xrightarrow{R} \mathrm{GL(V)}$ (where GL(V) are the general linear transformations on a vector space), such that $R(\mathbb{1}) = \mathbb{1}$ and $R(gh) = R(g)R(h)$. (In other words, R is a group homomorphism from G to GL(V).) Loosely speaking, we find matrices which represent the symmetry operators.
Examples: representations of the rotation group $SO(3)$

- trivial representation: on scalar quantities $R(\mathrm{Rot}) = \mathbb{1}$, no rotation.
- vector representation: on a vector, $V = \mathbb{R}^3$, $R(\mathrm{Rot}) = \mathrm{Rot}$. This is the so-called *fundamental representation*, in which the rotation matrices take the form that defines the group, namely $3x3$ orthogonal matrices with unit determinant.
- tensor representation: on a tensor, $V = \mathbb{R}^3 \times \mathbb{R}^3$, $R(\mathrm{Rot}) = \mathrm{Rot} \otimes \mathrm{Rot}$, because $T_{ij} \mapsto R_i{}^k R_j{}^l T_{kl}$.

Similarly, Lie algebras have representations and from a representation of a Lie algebra, we can always construct the associated representation of the group (by using the exponential map). Thus, we will sometimes be a bit sloppy and switch back and forth between algebra and group.
Let's make all of this more concrete by looking at the Lorentz and the Poincaré groups as our examples. These are our most important examples, because these encode the fundamental symmetries of spacetime, on which we are constructing our quantum field theory.

2.3 Lorentz transformations and the Lorentz group

Lorentz transformations, abstractly denoted by Λ, act on 4-vectors that denote the spacetime location of an event [3], i.e.,

$$x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} \tag{33}$$

as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{34}$$

where the defining equation for a Lorentz transformation is

$$\Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \eta_{\sigma\rho}. \tag{35}$$

This equation says that the Minkowski metric is left invariant under Lorentz transformations, which implies that scalar products built with this metric are invariant under Lorentz transformations.
Because η is the Minkowski metric, the Lorentz group is $SO(3,1)$, and contains boosts and spatial rotations, instead of being $SO(4)$, the group of rotations of 4-dimensional space (which Eq. (35) would define for $\eta_{\mu\nu} \to \delta_{\mu\nu}$).
From Eq. (35), we have that $\Lambda_\nu{}^\mu = \eta_{\mu\kappa}\eta_{\lambda\nu}\Lambda^\lambda{}_\kappa$ is the inverse Lorentz transformation. This is easiest to see by writing Eq. (35) in matrix notation, where it reads

$$\Lambda^T \eta \Lambda = \eta, \tag{36}$$

where the first Λ is transposed, in order for the index contraction in Eq. (35) to match index contraction for matrix multiplication. From Eq. (36), we then have that

$$\Lambda^{-1} = \eta^{-1} \Lambda^T \eta, \tag{37}$$

which, in index notation, becomes

$$\left(\Lambda^{-1}\right)^\mu{}_\nu = \eta^{\mu\kappa} \left(\Lambda^T\right)_\kappa{}^\lambda \eta_{\lambda\nu} = \eta^{\mu\kappa}\Lambda^\lambda{}_\kappa \eta_{\lambda\nu} = \Lambda_\nu{}^\mu. \tag{38}$$

When acting on 4-vectors, Λ are in their fundamental representation, which you can think of as the representation that is used to define the group. How is the associated Lie algebra $\mathfrak{so}(3,1)$ characterized?
We use that we can expand the exponential map to first order in the Lie algebra elements, if we consider an infinitesimal transformation. For the fundamental representation

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu + \mathcal{O}(\omega^2) \tag{39}$$

[3] Note that we use units in which $c = 1$.

for an infinitesimal transformation. Eq. (35) then implies a property of the ω's:

Mini-Exercise 3. What holds for $\omega^{\mu}{}_{\nu}$, such that (35) holds?

Solution.

$$\left(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho}\right)\eta_{\mu\nu}(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}) \stackrel{!}{=} \eta_{\sigma\rho}$$
$$\eta_{\rho\sigma} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + \mathcal{O}(\omega^2) = \eta_{\sigma\rho} \implies \omega_{\rho\sigma} = -\omega_{\sigma\rho}.$$

It holds that

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \tag{40}$$

i.e., $\omega_{\mu\nu}$ is an antisymmetric 4×4 matrix and therefore has 6 independent components that can be nonzero. Depending on which components we choose to be nonzero, we obtain a different group element of the Lorentz group.
Let us consider an example: We choose $\omega^{12} = -\omega^{21} = \theta$ and set all other components of ω to zero. Note that we have to be careful with the upper and lower indices on ω, so there will be an $\eta_{\mu\nu}$ that will make an appearance below. We obtain that

$$\begin{aligned} \Lambda^{\mu}{}_{\nu} &= \delta^{\mu}_{\nu} + \omega^{\mu\rho}\eta_{\rho\nu} \\ &= \mathbb{1} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta & 0 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{41}$$

We observe that this generates nothing but a (infinitesimal) rotation of the four-vector x^{μ} about the z-axis by an angle θ. We also note that the contraction $\omega^{\mu\rho}\eta_{\rho\nu}$ essentially flips the sign, i.e., $\omega^{1}{}_{2} = -\omega^{12}$ and similarly $\omega^{2}{}_{1} = -\omega^{21}$.
Similarly, if we choose $\omega^{01} = -\omega^{10} = \theta$, we obtain

$$\Lambda^{\mu}{}_{\nu} = \mathbb{1} + \begin{pmatrix} 0 & \theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{42}$$

which we can recognize as an infinitesimal boost along the x-axis, with θ being the rapidity, $\tanh\theta = v/c$. In this case, we have used that $\omega^{0}{}_{1} = \omega^{01}\eta_{11} = -\omega^{01}$ and $\omega^{1}{}_{0} = \omega^{10}\eta_{00} = -\omega^{10}$.

These examples help us to see that the six entries in $\omega^{\mu\nu}$ which can be nonzero, select, which among the six possible "basis" transformations (3 rotations along the 3 spatial axis, and 3 boosts along these axis), can be performed and by which amount the physical system is rotated and/or boosted. If we choose more than one component of $\omega^{\mu\nu}$ to be non-zero, we get the corresponding combination of these "basis" transformations.

For a general representation $U(\Lambda)$ of the Lorentz transformation Λ, we have that

$$U(\mathbb{1}+\omega) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + \mathcal{O}(\omega^2). \tag{43}$$

In this expression, the $\omega^{\mu\nu}$ still selects, which transformation is performed and determines the "amount" of the transformation, but the "basis transformations" are now encoded in the $M^{\mu\nu}$. The $M^{\mu\nu}$ are called the *generators* of the Lorentz group, and there are six of them, representing the three independent rotations and three independent boosts. We have that

$$\begin{aligned} J^i &= \frac{1}{2}\epsilon^i{}_{jk}M^{jk} && \text{generates rotations} \\ K^i &= M^{i0} && \text{generates boosts,} \end{aligned}$$

where ϵ_{ijk} is the Levi-Civita symbol, which is fully antisymmetric under permutations of its indices and $\epsilon_{123} = 1$. Thus, M^{12}, M^{13} and M^{23} generate rotations (along the z-, y- and x-axis, respectively) and M^{01}, M^{02} and M^{03} generate boosts along the x-axis, y-axis and z-axis, respectively.
A Lorentz transformation can act on many different objects, not just on four-vectors. In particular, we will later in the course encounter *spinors*, which are objects that have spinor indices. These are indices, i.e., a spinor is a collection of functions, but they are *not* spacetime indices. Therefore, to have a Lorentz transformation act on a spinor, the $M^{\mu\nu}$ need to carry the appropriate indices, i.e., *each of the six $M^{\mu\nu}$'s, such as M^{01}, M^{12} etc., must be a* matrix *with indices in the space that it acts on.*
This is somewhat abstract at this moment, so in order to make it less abstract, we consider the case in which the Lorentz transformation acts on a four-vector. We already know that we can write this in the form of Eq. (39), but now we want to understand how to write it in the form Eq. (43), in which the generators appear explicitly. In fact, for the fundamental representation of the Lorentz group, we have that

$$(M_{\mu\nu})_{\kappa\lambda} = -i\left(\eta_{\mu\kappa}\eta_{\nu\lambda} - \eta_{\nu\kappa}\eta_{\mu\lambda}\right). \tag{44}$$

By plugging this into Eq. (43), we get back Eq. (39).
While it seems unnecessarily complicated to introduce the M's for the action on 4-vectors, the main point about Eq. (43) is that it is general; it describes the action of a Lorentz transformation on any object.
The *defining property* of the generators of the Lorentz group is that they satisfy a commutation relation. The abstract definition of the Lie algebra of the Lorentz group is through this commutation relation:
The Lie algebra of the Lorentz group $SO(3,1)$, is defined by the commutator relation of its generators, which is

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}\,M^{\nu\sigma} - \eta^{\nu\rho}\,M^{\mu\sigma}) - i(\eta^{\mu\sigma}\,M^{\nu\rho} - \eta^{\nu\sigma}\,M^{\mu\rho}). \tag{45}$$

You will derive this commutation relation in the exercises. You can think of the Lorentz group as being defined by this commutation relation. When we talk about different elementary particles and different fields, they all arise from thinking about different representations of the Lorentz group, i.e., many properties of elementary particles follow from this commutation relation above. At this stage, this is still a rather abstract notion, but over the course of this course, we will see

the commutation relation Eq. (45) "unfold its power".

2.4 Poincaré group and why we classify particles by their mass and spin

We classify elementary particles by their mass and spin, plus quantum numbers associated to internal symmetries. For instance, we describe the electron as a particles with rest-mass 511 keV and spin-1/2 (and electric charge -1). Why do we do so? Is it just a conventional choice and we could be using some completely different characteristics? The answer is no. There is a deep mathematical reason and it has to do with the structure of the *Poincaré-group* - a generalization of the Lorentz group - and its so-called *Casimir-operators*.
The Poincaré group is an extension of the Lorentz group which, in addition to boosts and rotations, contains translations, under which $x^\mu \mapsto x^\mu + a^\mu$. A transformation by an element of the Poincaré group can be written as $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu + a^\mu$.
This is the full symmetry that 3+1-dimensional Minkowski spacetime enjoys.
An infinitesimal translation in a general representation can be written as

$$U(a) = \mathbb{1} + i a_\mu P^\mu, \tag{46}$$

where P^μ is the generator of translations. By Noethers theorem, P^μ will be identified as the 4-momentum in the corresponding representation. Its commutation relations with the other generators of the Poincaré group are

$$[P^\mu, M^{\rho\sigma}] = i(\eta^{\mu\sigma} P^\rho - \eta^{\mu\rho} P^\sigma) \tag{47}$$

$$[P^\mu, P^\nu] = 0. \tag{48}$$

Now let us consider some state of n particles, which transforms under actions of the Poincaré group. Under such transformations, its properties, such as its 4-momentum, change.
However, the Poincaré group has two *Casimir invariants*. These are (in the simplest case) quadratic combinations of generators, which commute with all other generators. Therefore, their eigenvalues are unchanged under the action of group elements and they provide *invariant* characterizations of particles.
$P^2 = P_\mu P^\mu$ is the first Casimir invariant and $W^2 = W_\mu W^\mu$, with $W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma$ the Pauli-Lubanski-pseudovector, is the second.

Mini-Exercise 4. Show that P^2 commutes with all generators of the Poincaré group.

Solution.

$$\begin{aligned}
[P^2, M_{\mu\nu}] &= [P^\rho P_\rho, M_{\mu\nu}] \\
&= P^\rho [P_\rho, M_{\mu\nu}] + [P^\rho, M_{\mu\nu}] P_\rho \\
&= P^\rho (i\eta_{\rho\nu} P_\nu - i\eta_{\rho\mu} P_\nu) + (i\eta_{\rho\nu} P_\mu - i\eta_{\rho\mu} P_\nu) P^\rho \\
&= i(P_\nu P_\mu - P_\mu P_\nu + P_\mu P_\nu - P_\nu P_\mu) \\
&= 0.
\end{aligned}$$

P^2 acting on a state with some 4-momentum yields the eigenvalue m^2, i.e., because P^2 is a Casimir operator of the Poincaré group, we label elementary particles by their rest mass.

But what is the physical meaning of W^2?

$$W^2 = W_\mu W^\mu = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma\epsilon^\mu{}_{\chi\lambda\tau}M^{\chi\lambda}P^\tau. \tag{49}$$

Let's consider this in a massive particles rest frame (massless particles are a separate case and we will get to them later).
Then $P \to (m, \vec{0})$ and $W^0 = 0$. This holds, because ϵ is totally antisymmetric and because the only non-zero component of P is P^0.

$$W^i = \frac{1}{2}\epsilon^i{}_{\mu\nu 0}M^{\mu\nu}P^0, \tag{50}$$

here μ, ν must be spatial indices, but $\neq i$. Therefore, $\epsilon^i{}_{jk0} = \epsilon^i{}_{jk}$, the 3d Levi-Civita symbol. Thus,

$$W^i = J^i P^0 = mJ^i \implies W^2 = -m^2\vec{J}\cdot\vec{J}. \tag{51}$$

Now we need to interpret which angular momentum it is that shows up here. Which angular momentum does an elementary particle have? *Intrinsic* angular momentum, i.e., spin. You might remember from QM, that the eigenvalues of $\vec{J}^2$ are $s(s+1)$, with s the spin.
$\implies$ Because W^2 is the 2nd Casimir operator of the Poincaré group, we label massive elementary particles by their spin.

We have thus come to our first concrete result from our more abstract consideration of group theory:
We have learned that there is a reason why we label elementary particles by mass and spin. This is not an arbitrary choice, but a direct consequence of the fundamental symmetry-structure of Minkowski spacetime and the properties of the underlying Poincaré group.

Next, we may wonder, what spin values[4] are allowed? Can we have elementary particles with spin 0? spin 1/2? Spin 1? What about non-half-integers? Is there a particle with spin 2/3? or spin M? To figure this out, we will classify the representations of the Lorentz group. This will determine what type of fields we will focus on for the rest of the course.[5] Generally, for a field with a general Lorentz index A (could be a 4-vector index, or two 4-vector indices, such that the field is a tensor, but we'll also encounter spinor indices, which label the components of a spinor, but are *not* spacetime indices), $\phi_a(x)$, we have

$$\phi'_a(x) = L_a{}^b(\Lambda)\phi_b(\Lambda^{-1}x). \tag{52}$$

[4] All in units of $\hbar$, which we set to 1.

[5] In the current discussion, we are switching back and forth between considering particles and fields. In this, we are already using a result that we will see a little later in the course, namely that elementary particles show up as excitations of fields. Therefore, it is to some extent equivalent to talk about particles or about the associated fields, because the properties of the particles follow from the properties of the fields. However, let us highlight that there is a difference when it comes to representations of the Poincaré group: fields transform in the finite-dimensional representations of the group, i.e., they are constructed from a finite set of components. In contrast, particles transform in the infinite-dimensional representation of the Poincaré group. Physically, this is, loosely speaking because if you have a particle with some four-momentum p^μ, then there are infinitely many other four-momenta $p^{\mu'}$ that are related to p^μ by a boost. The choice of an infinite-dimensional representation is also necessary, because no finite-dimensional representation is unitary, and we would like to have probabilities (or scalar products of a state with itself) to be preserved under Poincaré transformations. Therefore, the representation that a field transforms in is not the same one as the particles that it gives rise to transform in. However, for our purposes at the present, we do not yet need to know this, as we will now simply focus on the representations that the fields can transform in.

The matrices ${L_a}^b(\Lambda)$ form a *representation* of the Lorentz group, i.e.,

$$ {L_a}^b(1+\omega) = {\delta_a}^b + \frac{i}{2}\omega_{\mu\nu}{(M^{\mu\nu})_a}^b \tag{53} $$

where ${(M^{\mu\nu})_a}^b$ are representation matrices of the $\mathfrak{so}(3,1)$ Lie algebra, so that

$$ [M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\mu\rho}\, M^{\nu\sigma} - \eta^{\nu\rho}\, M^{\mu\sigma}) - i(\eta^{\mu\sigma}\, M^{\nu\rho} - \eta^{\nu\sigma}\, M^{\mu\rho}). \tag{54} $$

To understand which spins elementary particles can have, we must find all possible (finite-dimensional) matrices $M^{\mu\nu}_{ab}$ that obey these commutation relations, in order to find the possible fields that we can write down. This sounds like a challenging problem, but it turns out that we are lucky if we know something about the representation of the Lie algebra $SU(2)$ [6].
From QM, we know that $[J_i, J_j] = i\epsilon_{ijk}J_k$, which is the $SU(2)$ Lie algebra, is satisfied by sets of 3 hermitian matrices of size $(2j+1)\times(2j+1)$, where the eigenvalues of J_3 are $-j, -j+1, \ldots, +j$. (If you would like a "refresher" on this, a good place to read up on it is, e.g., Sakurai "Modern Quantum Mechanics".)

Our luck lies in the fact that upon introducing

$$ N_i = \frac{1}{2}(J_i - iK_i) \qquad (\text{remember}: J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} \text{ and } K_i = M_{i0}) \tag{55} $$

and

$$ M_i = \frac{1}{2}(J_i + iK_i) \tag{56} $$

(Note: J_i, K_i are hermitian; N_i is not; in fact $M_i = N_i^\dagger$.) we find that

$$ [N_i, N_j] = i\epsilon_{ijk}N_k, \qquad [M_i, M_j] = i\epsilon_{ijk}M_k, \qquad [N_i, M_j] = 0. \tag{57} $$

The Lie algebra of $SO(3,1)$ is nothing but two separate $SU(2)$ Lie algebras!
Thus, we can build the representations of the $SO(3,1)$ Lie algebra from representations of the $SU(2)$ Lie algebra!
$\Longrightarrow$ Each irreducible (i.e., not give by a product of two smaller representations) representation of the $SO(3,1)$ Lie algebra is specified by *two* integers or half-integers n' and n, which are the eigenvalues of M_3 and N_3.
We *label* these representations by n and n' or by the number of components in each representation, $(2n+1)$ and $(2n'+1)$.
To understand the corresponding spin of the field (and the particles that are the excitations of the field), we go back to the Pauli-Lubanski pseudovector and the associated Casimir operator, in the rest-frame, $W^2 = -m^2\vec{J}\cdot\vec{J}$ and also use that $J_i = M_i + N_i$. Thus,

[6] Note that the Lie algebras for $SO(3)$ and $SU(2)$ are identical. For the groups, there are some subtle differences, which need not directly concern us.

(n, n')	$\big(2n+1,\, 2n'+1\big)$	spin	name of the field
$(0,0)$	$(1,1)$	0	scalar (singlet)
$\big(\frac{1}{2},0\big)$	$(2,1)$	$\frac{1}{2}$	left-handed spinor
$\big(0,\frac{1}{2}\big)$	$(1,2)$	$\frac{1}{2}$	right-handed spinor
$\big(\frac{1}{2},\frac{1}{2}\big)$	$(2,2)$	1	vector (this has $2 \cdot 2 = 4$ components, which is the right number for a 4-vector)

Therefore, we now have a clear idea which fields we are going to consider. Rather than guessing that maybe there could be elementary particles with spin 2/3 (or other non half-integer values) out there, and somehow trying to come up with ideas for what the corresponding fields could be, we already know that such fields/particles do not exist and we do not need to spend our time trying to find a description for them, because our considerations, based on symmetries, tell us that such an effort is futile.

To sum up, by considering the fundamental symmetry of Minkowski spacetime, that a theory of fields and associated particles living on that spacetime has to satisfy, we have developed a comprehensive list of possible fields that can exist. Thus, rather than proceeding by trial-and-error, we have found a systematic structure that the rest of this course (and Quantum Field Theory) will follow. This structure is very restrictive and only allows us to consider fields which are associated to integer or half-integer spins. It is therefore not an accident that all elementary particles have integer or half-integer spin; there are no other options for them, based on the underlying symmetry group, the Poincaré group.

We will work our way through the spin 0, 1/2 and 1 cases in the course, because, as it turns out, they are all part of the Standard Model of particle physics.

Higher spins (3/2, 2) do not correspond to detected elementary particles, although spin 3/2 plays a role in supergravity, where a spin 3/2 particle is the superpartner of the graviton. The graviton, which is the expected quantum of the gravitational field, has spin 2.

2.5 Noether's theorem

Symmetries not only help us to understand the building blocks of our theory (i.e., which fields there may be and how we characterize particles), they also imply conserved quantities and thus determine dynamical processes. The link between symmetries and conserved quantities is at the heart of Noether's theorem, just as in classical mechanics.

Noether's theorem in QFT states that:
Every continuous symmetry of the action implies a conserved current density and a conserved charge.

This is similar to Noether's theorem in classical mechanics with the key difference being the conservation of the current. To derive the theorem, we will consider a scalar field; the theorem generalizes to non-zero spin fields, such as the gauge field and spinor fields.

As an example of a continuous spacetime symmetry, consider a translation $x \to x' = x + d$. How does $\phi \to \phi'$ look like? Note that we will take the *active point of view*, where we are assuming that the physical field configuration changes (in contrast to the *passive point of view*, where the coordinates change). It should hold that the transformed field at the transformed point is equal

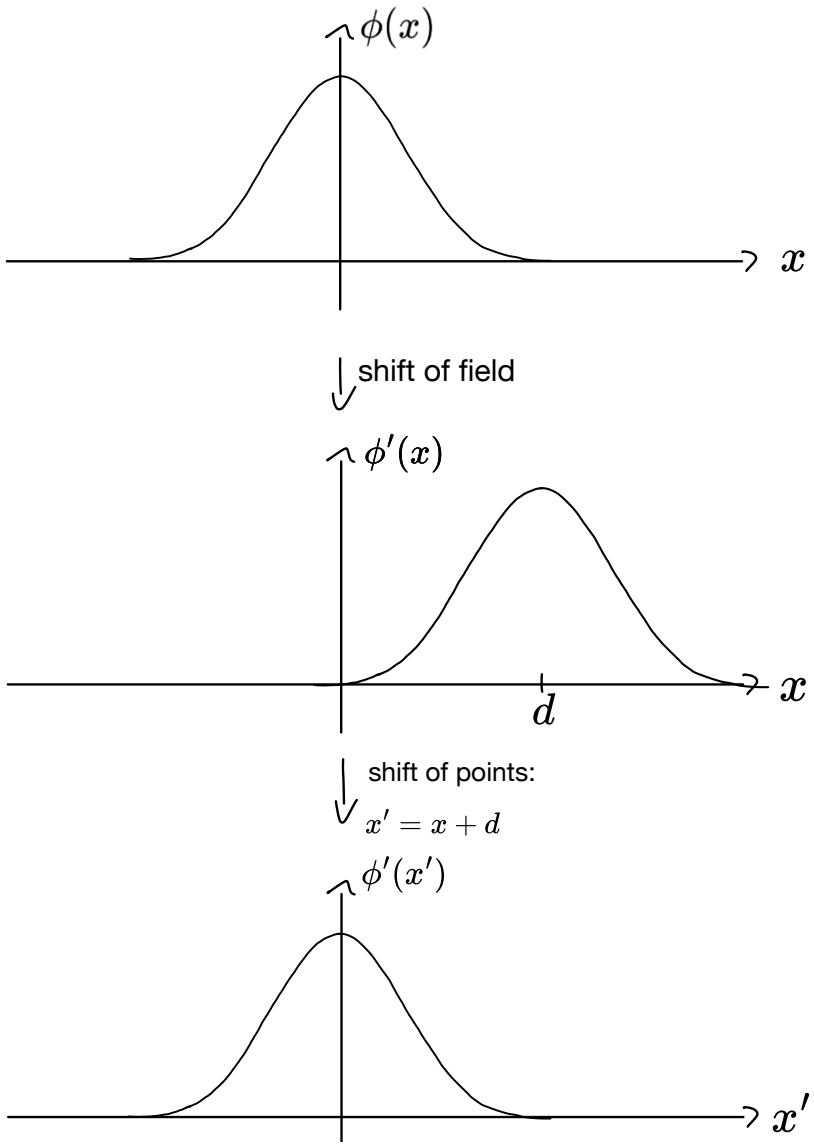


Figure 1: We show a field configuration $\phi(x)$ in the upper panel. In the central panel, we have shifted the field by a distance d (active transformation) and in the lower panel, we have then additionally shifted the coordinates by the same distance d, so that $x' = x + d$.

to the untransformed field at the original point, because, if we are shifting the field, but then also shift all points, the system remains unchanged. Thus

$$\phi'(x') = \phi(x), \tag{58}$$

which is shown in Fig. 1.
Thus, $\phi'(x)$ is defined by applying the inverse transformation to the argument, i.e.,

$$\phi'(x) = \phi(x - d). \tag{59}$$

When we generalize to a Lorentz transformation $x' = \Lambda x$, we have the same behavior: the scalar

field is evaluated at a point that corresponds to the inverse of the transformation.

$$\phi'(x) = \phi(\Lambda^{-1}x). \tag{60}$$

We can also consider *internal* symmetries, e.g., for a complex scalar field $\phi(x)$ taking values in $\mathbb{C}$ instead of in $\mathbb{R}$, we can write an action that has a $U(1)$ symmetry:

$$S_{\mathrm{U}(1)\,\mathrm{complex\,scalar}} = \int d^4x \left(\frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{1}{2}m^2\phi^*\phi\right), \tag{61}$$

which is invariant under $\phi(x) \to e^{i\alpha}\phi(x)$, and, accordingly $\phi^*(x) \to e^{-i\alpha}\phi^*(x)$. The infinitesimal version of this transformation is

$$\phi'(x) = \phi(x)\left(1 + i\alpha + ...\right). \tag{62}$$

We will consider this example in much more detail later in the lecture.
To derive Noether's theorem, we assume some continuous symmetry, but we do not need to specify whether it is a spacetime symmetry or an internal symmetry. Noether's theorem holds for both. Because we are assuming a continuous symmetry, there is an infinitesimal version of this transformation of the field

$$\phi(x) \to \phi'(x) = \phi(x) + \varepsilon\chi(x). \tag{63}$$

(For a discrete symmetry, there are only finite transformations, e.g. a $\mathbb{Z}_2$-symmetry under which $\phi(x) \to -\phi(x)$ has no infinitesimal version. This is why all that follows holds for continuous, but not for discrete symmetries.)

For instance, for an infinitesimal translation, we can write the right-hand side in terms of a Taylor expansion

$$\begin{aligned} \phi'(x) &= \phi(x) + \frac{\partial\phi}{\partial x^\mu}\, d^\mu + ..., & (64)\\ &= \phi(x) + \epsilon_\mu\chi^\mu(x), & (65) \end{aligned}$$

where we consider d^μ to be an infinitesimal shift and we defined $\partial_\mu\phi d^\mu = \epsilon_\mu\chi^\mu(x)$. When we perform a translation in a single direction in spacetime, this reduces back to the form $\epsilon_\mu\chi^\mu \to \epsilon\,\chi$. We denote the difference between the transformed and the untransformed field

$$\delta_\varepsilon\phi \coloneqq \phi' - \phi. \tag{66}$$

Under this change in the field, the Lagrangian changes as follows:

$$\begin{aligned} \delta_\varepsilon\mathcal{L} &= \mathcal{L}' - \mathcal{L} = \mathcal{L}(\phi', \partial\phi') - \mathcal{L}(\phi, \partial\phi) & (67)\\ &= \frac{\partial\mathcal{L}}{\partial\phi}\delta_\varepsilon\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\varepsilon\partial_\mu\phi, & (68) \end{aligned}$$

where $\delta_\varepsilon\partial_\mu\phi = \partial_\mu\phi' - \partial_\mu\phi$. (Note that we're slightly abusing naming conventions, as advertised, because this is the Lagrangian *density*, but we are referring to it as the Lagrangian. This is very common practise in QFT.)
Because we assume that the transformation corresponds to a symmetry of the action, the action

must stay invariant under it. Thus, the Lagrangian may at most change by a total derivative, so we can write

$$\delta_\varepsilon \mathcal{L} = \varepsilon \partial_\mu F^\mu(\phi, \partial\phi, \partial^2\phi, x), \tag{69}$$

where, depending on the symmetry F^μ may actually be zero, so that even the Lagrangian is invariant under the symmetry.
We know that $\delta_\varepsilon \mathcal{L} \sim \varepsilon$, because $\delta_\varepsilon \mathcal{L} \to 0$ for $\varepsilon \to 0$. In principle, $F^\mu = F^\mu(\phi, \partial\phi, \partial^2\phi, x)$ can have dependencies on x and on $\partial^2\phi$, etc., even if $\mathcal{L}$ does not.
Now we want to derive the conserved current. $\partial_\mu F^\mu$ is a good starting point, because it already has the required form for a conservation law, $\partial_\mu j^\mu = 0$.

$$\varepsilon \partial_\mu F^\mu = \delta_\varepsilon \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \partial_\mu \phi. \tag{70}$$

In the next step we use the equations of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0, \tag{71}$$

to rewrite the 1st term into a form that also has a partial derivative in front, as needed to derive a conservation law. Note that this will mean that everything that follows only applies for field configurations which satisfy the equations of motion. (In QFT, these are often called "on-shell" configurations. In a few weeks, when we talk about the path integral quantization, we will explicitly see the difference to the "off-shell" configurations.) We obtain

$$\implies \varepsilon \partial_\mu F^\mu = \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta_\varepsilon \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \partial_\mu \phi \tag{72}$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_\varepsilon \phi \right). \tag{73}$$

Thus,

$$\partial_\mu \Big(\underbrace{F^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi}_{j^\mu} \Big) = 0. \tag{74}$$

j^μ is a conserved current.

Example: Energy-momentum tensor and its conservation

In classical mechanics, the symmetry-transformation underlying energy-momentum conservation is a space-time-translation:

$$x^\mu \to x'^\mu = x^\mu - \epsilon^\mu. \tag{75}$$

(These are really 4 symmetries packaged into one.)
The resulting transformation of the field is, as we wrote above,

$$\phi'(x) = \phi(x + \varepsilon) \tag{76}$$

$$\implies \delta_\varepsilon \phi = \phi(x + \varepsilon) - \phi(x) = \varepsilon^\nu \underbrace{\partial_\nu \phi(x)}_{\chi_\nu} \tag{77}$$

Eq. (76) means that the new field at x is the same as the old one at $x+\varepsilon$, because the shift is by $-\varepsilon$, and we are again using the active view on transformations. In Eq. (77), χ has an index, because there are 4 symmetries. Eq. (77) is to 1st order in ε, because we can Taylor expand $\phi(x+\varepsilon)$. Thus, if we focus on the dependence of $\mathcal{L}$ on x (through its dependence on ϕ),

$$\mathcal{L}'(x) = \mathcal{L}(x+\varepsilon) \tag{78}$$

$$\implies \delta_\varepsilon \mathcal{L} = \mathcal{L}(x+\varepsilon) - \mathcal{L}(x) \tag{79}$$

$$= \varepsilon^\mu \partial_\mu \mathcal{L}(x) \qquad \text{(to 1st order in } \varepsilon) \tag{80}$$

$$= \varepsilon^\nu \partial_\mu (\underbrace{\delta^\mu{}_\nu \mathcal{L}}_{\coloneqq F^\mu{}_\nu}) \tag{81}$$

Now we can use the general expression we derived before to get the conserved currents. Because we are looking at 4 symmetries at the same time, we will have 4 conserved currents, each of which is a 4-vector. In Eq. (81), you can think of the index μ as the index that belongs to a conserved current (which is a four-vector) and the index ν as the one that labels the four distinct currents that there are for the four distinct translations. Which translation is performed, is selected by the non-zero components of ϵ^ν.
To "package" the four conserved currents into one expression, we write

$$j^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \chi_\nu - F^\mu{}_\nu \tag{82}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L}. \tag{83}$$

This conserved *tensor* is usually written as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}. \tag{84}$$

It is conserved, $\partial_\mu T^{\mu\nu} = 0$, and called the energy-momentum (or stress-energy) tensor.
Let us clarify the status of $T^{\mu\nu}$: we could derive the expression $j^{\mu\nu}$ in Eq. (83) from spacetime translations even if spacetime translations would not correspond to a symmetry of the action. However, we would *not* have that $j^{\mu\nu}$ is a conserved quantity. In other words, if $S' \neq S$ under a transformation, we will have $\partial_\mu j^\mu \neq 0$ for the j^μ corresponding to this transformation.

Side-note: The energy-momentum tensor is an entry-point into General Relativity, because, if we promote $\eta^{\mu\nu} \to g^{\mu\nu}$, then $T^{\mu\nu}$ acts as a source for spacetime curvature in the Einstein equations. The physical meaning behind that is that any form of energy or momentum sources spacetime curvature.

In our statement of Noether's theorem, we also mentioned the conservation of a charge. Let us derive what the conserved charge is. From the conservation of the current, we can also derive the conservation of a charge:

$$Q(t) = \int d^3x \, j^0(t, \vec{x}). \tag{85}$$

It holds that

$$\dot{Q} = \frac{\mathrm{d}}{\mathrm{d}t} Q(t) = 0, \tag{86}$$

if we assume that all fields and their derivatives vanish at $|x| \to \infty$, i.e., we only consider nonzero field configurations away from spatial infinity. This is reasonable to describe all realistic physical situations that we are interested in (e.g., particle physics experiments at CERN, phonons in the Bose-Einstein-Condensates of our experimental friends in Neuenheimer Feld, or superconducting Cooper-pairs in superconductors in various labs, all of which are described by QFT.)
We can show $\dot{Q} = 0$ as follows:

$$\dot{Q} = \frac{\mathrm{d}}{\mathrm{d}t} \int d^3x\, j^0(t,\vec{x}) = \int d^3x \big(\partial_0 j^0(t,\vec{x})\big) \tag{87}$$

$$= -\int d^3x\, \partial_i j^i(t,\vec{x}) \qquad \text{(by conservation of the current)} \tag{88}$$

$$= -\int dxdydz(\partial_x j^x + \partial_y j^y + \partial_z j^z) \tag{89}$$

$$= -\int dydz\, j^x\Big|_{x\to\pm\infty} + \int dxdz\, j^y\Big|_{y\to\pm\infty} + \int dxdy\, j^z\Big|_{z\to\pm\infty} \tag{90}$$

$$= 0, \tag{91}$$

if fields and derivatives vanish at $|x| \to \infty$, so that j vanishes there.
Let us highlight that the conservation of a *current* is stronger than the conservation of the charge, because it implies that the charge is conserved *locally*, i.e., changes of the charge in a (finite) volume in time must be accounted for by a current flowing though the surface of the volume. To see this, write:

$$\frac{\mathrm{d}Q_V}{\mathrm{d}t} = -\int_V d^3x\, \vec{\nabla}\cdot\vec{j} = -\int_{A=\partial V} \vec{j}\cdot d\vec{S}. \tag{92}$$

Q_V is the charge in a volume V. In the last step we used Gauss' theorem for volume integrals of divergences.
Example: the conserved charges following from the conservation of the energy-momentum tensor are:

$$\int d^3x\, T^{00} = \underbrace{\int d^3x \left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} - \mathcal{L}\right)}_{\substack{\text{We recognize this}\\ \text{as the Hamiltonian!}}} = H =: P^0 \tag{93}$$

The other conserved charges are the spatial momenta, so $P^\nu = \int d^3x\, T^{0\nu}$ is conserved.

Noether's theorem also applies to continuous internal symmetries. We'll consider an example later in the course.

Mini-Exercise 5. Take

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{94}$$

What is P^i?

Solution.

$$P^i = \int d^3x\, T^{0i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial^i \phi - \underbrace{\eta^{0i}}_{=0} \mathcal{L}$$
$$= \dot{\phi}\, \partial^i \phi. \qquad \text{(not surprisingly only dependent on kinetic energy)}$$

We note that the conserved quantities in turn are the *generators* of the associated symmetry. This closes our considerations of symmetries. We have learned that symmetries are encoded in groups. Continuous symmetries of interest in physics are Lie groups, for which each symmetry transformation can be generated by the generators of the Lie algebra. In turn, Noether's theorem tells us that each symmetry leads to a conserved quantity. This conserved quantity is the generator that generates this symmetry.
If we did not know about the Lie group associated to a symmetry, we could therefore learn about it from the action of the symmetry and the resulting conserved quantities.

3 Canonical quantization of the free scalar field

We are now ready to quantize the scalar field. We will first quantize it according to canonical quantization, because that makes the connection between fields and particles clear. We will then quantize it according to path-integral quantization, because that makes the quantum superposition principle clear. By introducing both canonical and path-integral formalism early on (as not all QFT books do), we are also acquiring a versatile toolbox with which to build more complicated QFTs (e.g., the Standard Model of particle physics) later. In addition, the canonical framework and the path-integral framework provide us with a different type of intuition about quantum fields, so it is very useful to know both.

3.1 Canonical Quantization of the real scalar field in the Schrödinger picture

We start with the non-interacting field, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. In the Schrödinger picture, we only consider the spatial dependence of the field (the time-dependence will be carried by the states).
We work in a formal analogy to quantum mechanics, where we generalize its formal structure. In Quantum Mechanics, we have the operators x and $p = \frac{dL}{d\dot{x}}$, for which we impose $[x, p] = i$ (in units where $\hbar = 1$). We will generalize this structure by making the analogy $x \simeq \phi(\vec{x})$ and $p \simeq \Pi(\vec{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$.[7]
In analogy to quantum mechanics, we promote the field and its conjugate field to operators[8] and demand commutation relations:

$$[\phi(\vec{x}), \Pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{95}$$

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 = [\Pi(\vec{x}), \Pi(\vec{y})]. \tag{96}$$

The operators have no time dependence, because we work in the Schrödinger picture, where states carry time-dependence and operators do not. The operators depend on the spatial position, $\vec{x}$, because they are fields and thus describe infinitely many degrees of freedom (one at each spacetime point) rather than finitely many, as in QM.

So far, these are formal expressions. How can we build the Hilbert space of states and what operators will the states be eigenstates of? Also, how does the field act on the states; what is the physical meaning of that?
To answer all of these questions, we start from the observation that the Hamiltonian H is very reminiscent of an (infinite) set of harmonic oscillators (one for each point) and we know how to quantize the harmonic oscillator. In detail, the Hamiltonian of the free scalar field is:

$$H = \frac{1}{2}\int d^3x\, (\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2) \tag{97}$$

In comparison, the Hamiltonian of the harmonic oscillator in quantum mechanics is $H_{\text{QM}} = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$.

[7] Note that $\Pi(\vec{x})$ has nothing to do with the physical momentum, i.e., the momentum $\vec{P}$ that we can derive as a conserved quantity from Noether's theorem for spatial translation. The physical momentum can be expressed in terms of the fields, as is done at the end of the last chapter.

[8] Note: We are *not* putting hats on operators. It should be clear from the context, when we are dealing with the QM theory in its canonical formulation and hence with operators.

However, the structure of H is not fully analogous to $H_{\rm QM}$, because the term $\sim (\vec{\nabla}\phi^2)$ couples harmonic oscillators at neighboring points.
We can actually decouple the oscillators by going to Fourierspace, where $(\vec{\nabla}\phi)^2$ will become $\sim (\vec{p}\phi)^2$ and H will become an infinite set of *decoupled* harmonic oscillators, one for each Fourier mode (or momentum).
We write the field $\phi(\vec{x})$ in terms of its Fourier modes $\tilde{\phi}(\vec{p})$ as

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(\vec{p}) \tag{98}$$

and the inverse relation

$$\tilde{\phi}(\vec{p}) = \int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\phi(\vec{x}), \tag{99}$$

where, to show this inverse relation, it is crucial that

$$\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^3(\vec{x}-\vec{y}). \tag{100}$$

You will prove this representation of the Dirac delta-distribution in the exercises.
Similar expressions hold for $\Pi(\vec{x})$. From these, and the commutators for $\phi(\vec{x})$ and $\Pi(\vec{y})$, we can deduce $\big[\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})\big]$.

Mini-Exercise 6. Deduce what the commutator $\big[\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})\big]$ is.

Solution.

$$\begin{aligned}\big[\tilde{\phi}(\vec{p}), \tilde{\Pi}(\vec{q})\big] &= \int d^3x \int d^3y\, e^{-i\vec{p}\cdot\vec{x}} e^{-i\vec{q}\cdot\vec{y}} \underbrace{[\phi(\vec{x}), \Pi(\vec{y})]}_{i\delta^3(\vec{x}-\vec{y})} \\ &= i\int d^3x\, e^{-i\vec{x}\cdot(\vec{p}+\vec{q})} \\ &= i(2\pi)^3\delta^3(\vec{p}+\vec{q}).\end{aligned}$$

Further, one can similarly check that

$$\big[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})\big] = 0 = \big[\tilde{\Pi}(\vec{p}), \tilde{\Pi}(\vec{q})\big]. \tag{101}$$

To evaluate H in Fourierspace, let's first focus on the term

$$\int d^3x\, (\vec{\nabla}\phi)^2 = \int d^3x \int \frac{d^3p}{(2\pi)^3}\, i\vec{p}\, e^{i\vec{p}\cdot\vec{x}}\tilde{\phi}(\vec{p}) \int \frac{d^3q}{(2\pi)^3}\, i\vec{q}\, e^{i\vec{q}\cdot\vec{x}}\tilde{\phi}(\vec{q}). \tag{102}$$

The scalar product between the two gradients, $\vec{\nabla}\cdot\vec{\nabla}$ becomes the scalar product between $\vec{p}$ and $\vec{q}$. We proceed by rearranging the terms so that all those involving x are grouped together in the

back:

$$\int d^3x\,(\vec{\nabla}\phi)^2 = \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\,(-\vec{p}\cdot\vec{q})\tilde{\phi}(\vec{p})\tilde{\phi}(\vec{q})\underbrace{\int d^3x\, e^{i\vec{p}\cdot\vec{x}}e^{i\vec{q}\cdot\vec{x}}}_{(2\pi)^3\delta^3(\vec{p}+\vec{q})} \tag{103}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,(+\vec{p}^2)\tilde{\phi}(\vec{p})\tilde{\phi}(-\vec{p}). \tag{104}$$

$\tilde{\phi}(-\vec{p})$ is related to $\tilde{\phi}(\vec{p})$ through complex conjugation, because $\phi(\vec{x})$ is real:

$$\phi(\vec{x}) = \phi^*(\vec{x}) \implies \tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p}). \tag{105}$$

We write a dagger, because we are dealing with operators. The classical field satisfies $\phi^*(\vec{p}) = \phi(-\vec{p})$. Overall, we arrive at

$$H = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\Big(\underbrace{\left|\tilde{\Pi}\right|^2}_{\tilde{\Pi}\cdot\tilde{\Pi}^\dagger} + \underbrace{(\vec{p}^2+m^2)}_{=\omega_{\vec{p}}^2}\left|\tilde{\phi}\right|^2\Big). \tag{106}$$

Now we have achieved full analogy with an (infinite, but not coupled) set of harmonic oscillators and so we know how to proceed to construct the Hilbert space!

At a conceptual level, the expression highlights

- the difference to QM: one (or finitely many) vs. infinitely many degrees of freedom
- a dangerous assumption: We assumed in this derivation, that this theory is consistent up to arbitrarily high energies (arbitrarily high $\omega_{\vec{p}}$). We will recap the consequences when we come to the topic of Ultraviolet (UV) divergences later.
  Note: UV has nothing to do with a frequency of light here; in QFT, it refers to "high energy".

3.2 Creation and annihilation operators and construction of the Fock space

We have rewritten the Hamiltonian for the non-interacting scalar field as an infinite set of decoupled harmonic oscillators, one for each momentum $\vec{p}$.
Conceptually, this highlights the difference between QM and QFT: finitely vs. infinitely many degrees of freedom (we will see how they are related to particles).
Mathematically, it means that we can follow the procedure to solve the simple harmonic oscillator in QM and generalize to QFT.

For the simple harmonic oscillator in QM, with Hamiltonian $H_{\text{QM}} = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2$, we introduce $a = \frac{1}{2}(\sqrt{2\omega\, m}\,x + i\sqrt{\frac{2}{\omega m}}\,p)$ and obtain $H = \omega(a^\dagger a + \frac{1}{2})$ and $\left[a, a^\dagger\right] = 1$ and can construct all eigenstates of H with the "ladder" operators a, $a^\dagger$.
In analogy, we define an operators *a for each $\vec{p}$, i.e., one simple harmonic oscillator for each of the infinitely many values of the spatial momentum $\vec{p}$*, and check whether this works out. (Note that this is not a priori clear, because $\tilde{\phi}(\vec{p})$ is conjugate to $\tilde{\Pi}(-\vec{p})$ and not $\tilde{\Pi}(\vec{p})$, because $\left[\tilde{\phi}(\vec{p}), \tilde{\Pi}(-\vec{q})\right] =$

$i(2\pi)^3\delta^3(\vec{p}-\vec{q})$.) We define:

$$a_{\vec{p}} = \frac{1}{2}\Big(\sqrt{2\omega_{\vec{p}}}\,\tilde{\phi}(\vec{p}) + i\sqrt{\frac{2}{\omega_{\vec{p}}}}\,\tilde{\Pi}(\vec{p})\Big) \tag{107}$$

The form of $a^\dagger_{\vec{p}}$ follows:

$$a^\dagger_{\vec{p}} = \frac{1}{2}\Big(\sqrt{2\omega_{\vec{p}}}\,\tilde{\phi}(-\vec{p}) - i\sqrt{\frac{2}{\omega_{\vec{p}}}}\,\tilde{\Pi}(-\vec{p})\Big), \tag{108}$$

where we used that $\tilde{\phi}^\dagger(\vec{p}) = \tilde{\phi}(-\vec{p})$ and $\tilde{\Pi}^\dagger(\vec{p}) = \tilde{\Pi}(-\vec{p})$. In the next step, we have to figure out the commutation relations of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$ from those for the field and its conjugate field.

Mini-Exercise 7. What is $\Big[a_{\vec{p}}, a^\dagger_{\vec{q}}\Big]$? Work efficiently and don't write out terms that will be zero.

Solution. By using $a_{\vec{p}}$ expressed through $\tilde{\phi}$, $\tilde{\Pi}$, we can rewrite $\Big[a_{\vec{p}}, a^\dagger_{\vec{q}}\Big]$ in terms of the commutators of $\tilde{\phi}$ and $\tilde{\Pi}$. By using that $\big[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{q})\big] = 0 = \big[\tilde{\Pi}(\vec{p}), \tilde{\Pi}(\vec{q})\big]$, we can directly reduce this to

$$\begin{aligned}
\Big[a_{\vec{p}}, a^\dagger_{\vec{q}}\Big] &= \left[\frac{1}{2}\sqrt{2\omega_{\vec{p}}}\,\tilde{\phi}(\vec{p}), \frac{-i}{2}\sqrt{\frac{2}{\omega_{\vec{q}}}}\tilde{\Pi}(-\vec{q})\right] + \left[\frac{1}{2}i\sqrt{\frac{2}{\omega_{\vec{p}}}}\tilde{\Pi}(\vec{p}), \frac{1}{2}\sqrt{2\omega_{\vec{q}}}\,\tilde{\phi}(-\vec{q})\right] \\
&= \frac{-i}{2}\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}i(2\pi)^3\delta^3(\vec{p}-\vec{q}) + \frac{i}{2}\sqrt{\frac{\omega_{\vec{q}}}{\omega_{\vec{p}}}}i(2\pi)^3(-)\delta^3(-\vec{q}+\vec{p}) \\
&= (2\pi)^3\delta^3(\vec{p}-\vec{q}).
\end{aligned}$$

In the last step, we set $\sqrt{\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}}}\delta^3(\vec{p}-\vec{q}) = \delta^3(\vec{p}-\vec{q})$, because, wherever the prefactor $\frac{\omega_{\vec{p}}}{\omega_{\vec{q}}} \neq 1$, $\delta^3(\vec{p}-\vec{q}) \to 0$.

Similarly, we can derive that

$$[a_{\vec{p}}, a_{\vec{q}}] = 0 = \Big[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\Big]. \tag{109}$$

Thus, we have established analogous commutations relations between the a's and $a^\dagger$'s as those in QM. Now, we want to check whether we can also rewrite the Hamiltonian in a way that is analogous to QM. For that, it is useful to write $\tilde{\phi}(\vec{p})$ and $\tilde{\Pi}(\vec{p})$ in terms of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$.
From (107) and (108), we get

$$\tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}}) \tag{110}$$

$$\tilde{\Pi}(\vec{p}) = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}(a_{\vec{p}} - a^\dagger_{-\vec{p}}). \tag{111}$$

Thus, the Hamiltonian becomes

$$H = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\Big(\big|\tilde{\Pi}\big|^2 + \omega_{\vec{p}}^2|\tilde{\phi}|^2\Big) \tag{112}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2}\Big(\frac{-i\omega_{\vec{p}}}{2}(a_{\vec{p}} - a^\dagger_{-\vec{p}})i(a^\dagger_{\vec{p}} - a_{-\vec{p}}) + \omega_{\vec{p}}^2\frac{1}{2\omega_{\vec{p}}}(a_{\vec{p}} + a^\dagger_{\vec{p}})(a^\dagger_{\vec{p}} + a_{-\vec{p}})\Big) \tag{113}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_{\vec{p}}}{4}\Big(a_{\vec{p}}a^\dagger_{\vec{p}} - \cancel{a_{\vec{p}}a_{-\vec{p}}} - \cancel{a^\dagger_{-\vec{p}}a^\dagger_{\vec{p}}} + a^\dagger_{-\vec{p}}a_{-\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}} + \cancel{a_{\vec{p}}a_{-\vec{p}}} + \cancel{a^\dagger_{-\vec{p}}a^\dagger_{\vec{p}}} + a^\dagger_{-\vec{p}}a_{-\vec{p}}\Big) \tag{114}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_{\vec{p}}}{2}(a_{\vec{p}}a^\dagger_{\vec{p}} + \underline{a^\dagger_{-\vec{p}}a_{-\vec{p}}}). \tag{115}$$

In the underlined term, we rename the integration variable $\vec{p} \to -\vec{p}$. Under this change, $\omega_{\vec{p}} = \omega_{-\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ and $\int d^3p \to \int d^3p$. Thus,

$$H = \int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_{\vec{p}}}{2}(a_{\vec{p}}a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}}a_{\vec{p}}). \tag{116}$$

Just like in QM, it will be useful to rewrite the order or $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$, which we can do by using the commutator.

$$\Rightarrow H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\,\Big(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}\underbrace{\Big[a_{\vec{p}}, a^\dagger_{\vec{p}}\Big]}_{(2\pi)^3\delta^3(0)}\Big). \tag{117}$$

We interpret $(2\pi)^3\delta^3(0) = \int d^3x\, e^{i\vec{0}\cdot\vec{x}} = \int d^3x\ = \mathrm{Vol}(\mathbb{R}^3) =: V$ as the volume of space, so that

$$H = \int \frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\,\left(a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{V}{2}\right). \tag{118}$$

The part $\int \frac{d^3p}{(2\pi)^3}\,\omega_{\vec{p}}\frac{V}{2}$ is *divergent*, due to the contribution of zero-point energies of harmonic oscillators with arbitrarily high frequency (even if V is kept finite). This is called an ultraviolet (UV) divergence.

We will encounter more UV divergences soon, when we will discuss *regularization* and *renormalization*. The physical reason behind UV divergences is always that we consider momentum integrals to arbitrarily high momenta. In doing so, we are assuming that the theory that we are considering holds to arbitrarily small distances (high momenta). This is clearly a (wild) extrapolation, because experimentally we can only check QFTs (e.g., those that make up the Standard Model) down to $\sim 10^{-19}$m (or 10TeV). In QFTs relevant to condensed matter, there is a known UV cutoff in the form of the lattice spacing between atoms. Below this cutoff, it does not make sense to think about, e.g., phonons or other collective excitations that are described by a QFT.

Specifically, the UV divergence associated to the *vacuum energy*, $V\int \frac{d^3p}{(2\pi)^3}\,\frac{\omega_{\vec{p}}}{2}$, can be ignored in QFT on $\mathbb{M}^4$ in the absence of non-trivial boundary conditions, because we can only *measure* differences in energy, but not a constant, ever present (even if infinite) shift in H.

However, once we couple QFT to gravity, the vacuum energy *curves* spacetime and acts as a *cosmological constant*. The fact that the physical, finite value (after renormalization) of it cannot be calculated, but is a free parameter, is part of the cosmological-constant problem.

Now we can continue with constructing the Fock space. To do so, we postulate a vacuum state $|0\rangle$,

$$a_{\vec{p}}\,|0\rangle = 0 \qquad \forall\,\vec{p}. \tag{119}$$

One-particle states are defined as

$$|\vec{p}\rangle = a^\dagger_{\vec{p}}\,|0\rangle\,. \tag{120}$$

Why is this called a one-particle state?

Mini-Exercise 8. Calculate $H\,|\vec{p}\rangle$ and from your finding, explain why $|\vec{p}\rangle$ is called one particle state.

Solution.

$$\begin{aligned} H\,|\vec{p}\rangle &= \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} a^\dagger_{\vec{p}}\,|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3}\,\omega_{\vec{k}} a^\dagger_{\vec{k}} \underbrace{\left[a_{\vec{k}}, a^\dagger_{\vec{p}}\right]}_{(2\pi)^3\delta^3(\vec{k}-\vec{p})}\,|0\rangle + 0 \qquad (\text{using } a_{\vec{p}}\,|0\rangle = 0) \\ &= \omega_{\vec{p}}\, a^\dagger_{\vec{p}}\,|0\rangle \\ &= \omega_{\vec{p}}\,|\vec{p}\rangle\,. \end{aligned}$$

$\omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ is the relativistic energy of a single particle of mass m and momentum $\vec{p}$. This justifies calling $|\vec{p}\rangle$ a one-particle state.

Note that this state contains a single particle of fixed momentum, but completely delocalized. We will look at localized states later.
We can continue with N-particle states

$$|\vec{p}_1 \ldots \vec{p}_N\rangle = a^\dagger_{\vec{p}_1} \ldots a^\dagger_{\vec{p}_N}\,|0\rangle \tag{121}$$

and find

$$H\,|\vec{p}_1 \ldots \vec{p}_N\rangle = (\omega_{\vec{p}_1} + \ldots + \omega_{\vec{p}_N})\,|\vec{p}_1 \ldots \vec{p}_N\rangle\,. \tag{122}$$

So far, these states are not normalized yet. We impose

$$||0\rangle|^2 = \langle 0|0\rangle = 1. \tag{123}$$

Then,

$$\left(a^\dagger_{\vec{p}}\,|0\rangle\right) \cdot \left(a^\dagger_{\vec{q}}\,|0\rangle\right) = \langle 0|\, a_{\vec{p}}\, a^\dagger_{\vec{q}}\,|0\rangle = (2\pi^3)\delta^3(\vec{p} - \vec{q}), \tag{124}$$

i.e., one-particle states with different momenta are orthogonal. However, we want to change the normalization in order to account for *Lorentz covariance*.
We want

$$\Lambda\,|p\rangle = \big|p'\big\rangle\,, \qquad \text{if } p'^\mu = \Lambda^\mu{}_\nu p^\nu, \tag{125}$$

such that we can demand

$$\big\langle p'\big|q'\big\rangle = \langle p|q\rangle\,, \tag{126}$$

i.e., a norm that does not depend on the Lorentz frame. Relativistically normalized states, denoted by $|p\rangle$ to distinguish them from $|\vec{p}\rangle$, are

$$|p\rangle = \sqrt{2\omega_{\vec{p}}}\, a^\dagger_{\vec{p}}\,|0\rangle\,. \tag{127}$$

This result will be derived in the exercises; the corresponding derivation will be added here afterwards.
To find the appropriate normalization, we write

$$\int d^4p \;\; \delta^4(\omega_{\vec{p}}^2 - \vec{p}^{\,2} - m^2)\Big|_{\omega_{\vec{p}}>0} = \int \frac{d^3p}{2\omega_{\vec{p}}}. \tag{128}$$

The left hand side (lhs) is Lorentz -invariant, because the sign of $\omega_{\vec{p}}$ does not change under boosts. In going to the right hand side (rhs), we use

$$\delta(f(x)) = \sum_i \frac{\delta(x-a_i)}{\left|\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{a_i}\right|}, \tag{129}$$

where a_i are the zeros of f.
$\omega_{\vec{p}}^2 - \vec{p}^{\,2} - m^2$ has solutions $\omega_{\vec{p}} = \pm\sqrt{\vec{p}^{\,2}+m^2}$. However, the minus sign is not a viable zero given our requirement $\omega_{\vec{p}} > 0$.

$$\Longrightarrow \int d^4p \;\; \delta(\omega_{\vec{p}}^2 - \vec{p}^{\,2} - m^2)\Big|_{\omega_{\vec{p}}>0} = \int d^4p \;\; \frac{\delta(\omega_{\vec{p}} - \sqrt{\vec{p}^{\,2}+m^2})}{2\omega_{\vec{p}}}\Big|_{\omega_{\vec{p}}>0} = \int \frac{d^3p}{2\omega_{\vec{p}}}. \tag{130}$$

$\Longrightarrow 2\omega_{\vec{p}}\,\delta^3(\vec{p}-\vec{q})$ is Lorentz invariant, because

$$\underbrace{\int \frac{d^3p}{2\omega_{\vec{p}}}}_{\substack{\text{Lorentz-inv.,}\\ \text{see above}}} \cdot 2\omega_{\vec{p}}\,\delta^3(\vec{p}-\vec{q}) = \underbrace{1}_{\text{Lorentz-inv.,}}. \tag{131}$$

$\Longrightarrow$ Relativistically normalized states, denoted by $|p\rangle$ to distinguish them from $|\vec{p}\rangle$, are

$$|p\rangle = \sqrt{2\omega_{\vec{p}}}\, a_{\vec{p}}^\dagger |0\rangle. \tag{132}$$

In summary, a consistent set of creation/annihilation operators, field expansion and states is given by

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}}e^{-ipx} + a_{\vec{p}}^\dagger e^{ipx}\right), && (133)\\
\Pi(x) &= (-i)\int \frac{d^3p}{(2\pi)^3}\frac{\sqrt{\omega_{\vec{p}}}}{\sqrt{2}}\left(a_{\vec{p}}e^{-ipx} - a_{\vec{p}}^\dagger e^{ipx}\right), && (134)\\
\left[a_{\vec{p}}, a_{\vec{p}'}^\dagger\right] &= (2\pi)^3\delta^3(\vec{p}-\vec{p}'), && (135)\\
|p\rangle &= \sqrt{2\omega_{\vec{p}}}\, a_{\vec{p}}^\dagger |0\rangle, && (136)\\
\langle p'|p\rangle &= (2\pi)^3\, 2\omega_{\vec{p}}\,\delta^3(\vec{p}-\vec{p}') && (137)
\end{aligned}$$

Alternatively, we can change the normalization of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ by

$$a_{\vec{p}} \to \left(\sqrt{2\omega_{\vec{p}}}\right)^{-1} a_{\vec{p}}, \quad a_{\vec{p}}^\dagger \to \left(\sqrt{2\omega_{\vec{p}}}\right)^{-1} a_{\vec{p}}^\dagger. \tag{138}$$

This does of course not change our normalization of the scalar product of states, because this is independent of the definition of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ (and should of course be Lorentz invariant in any

normalization of the operators – this property can also not change when we simply absorb factors into our definition of the creation and annihilation operators). In this other normalization, we accordingly have

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}\left(a_{\vec{p}}e^{-ipx}+a^\dagger_{\vec{p}}e^{ipx}\right), && (139)\\
\Pi(x) &= (-i)\int \frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}e^{-ipx}-a^\dagger_{\vec{p}}e^{ipx}\right), && (140)\\
\left[a_{\vec{p}},a^\dagger_{\vec{p}'}\right] &= (2\pi)^3\,2\omega_{\vec{p}}\,\delta^3(\vec{p}-\vec{p}'), && (141)\\
|p\rangle &= a^\dagger_{\vec{p}}|0\rangle, && (142)\\
\langle p'|p\rangle &= (2\pi)^3\,2\omega_{\vec{p}}\,\delta^3(\vec{p}-\vec{p}') && (143)
\end{aligned}$$

Depending on the book that you are consulting, you may find one or the other normalization. We may also sometimes switch between different normalizations throughout these lecture notes.

3.3 Casimir effect

A recurring theme throughout this course will be the result that quantum fluctuations have non-trivial physical consequences. Examples that we will consider later include the result that quantum fluctuations can spontaneously break a symmetry that the classical theory has (Coleman-Weinberg potential) and that quantum fluctuations can produce interactions between photons, even though electromagnetic waves are non-interacting in the classical theory (Euler-Heisenberg Lagrangian). A second, related theme is that in the corresponding calculations, divergences show up. Historically, these led (and actually still lead) to confusion, because it is not always straightforward to see that these divergences occur in unphysical, un-measurable quantities and the physical, measurable quantities are finite. The divergences are treated through regularization and renormalization.

In nature, the Casimir effect occurs for the electromagnetic field (that we will quantize in a few weeks). It is a result of the fact that on a conducting plate, the electric field must vanish (because otherwise it induces a current that counteracts the field). Thus, if we place two parallel, conducting plates in a vacuum, the field has to satisfy boundary conditions, namely that it vanishes at the location of both plates:

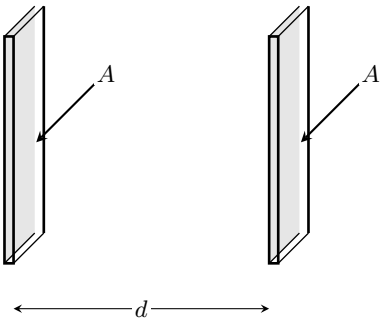


Figure 2: Two parallel plates at distance d, each of area A.

We model this effect with a massless scalar field and simplify the situation to 1+1 dimensions, so

that we can forget about the directions parallel to the plates. Then, we impose boundary conditions

$$\phi(0) = 0 = \phi(d). \tag{144}$$

From this, it follows that the momentum is discrete, allowed values are

$$p_x = \frac{n\pi}{d}, \tag{145}$$

so that

$$\omega_{\vec{p}}^2 = \vec{p}^2 \to \frac{n\pi}{d}, \tag{146}$$

and the integral $\int \frac{dp_x}{2\pi} \to \frac{1}{d}\sum_{n=1}^{\infty}$.
Thus, the expression for the ground-state energy density is

$$\epsilon = \frac{\langle 0|H|0\rangle}{d} = \frac{1}{d^2}\sum_n \frac{n\pi}{d}\left(\langle 0|a_{\vec{p}}^\dagger a_{\vec{p}}|0\rangle + \frac{d}{2}\right) = \frac{1}{2d}\sum_n \frac{n\pi}{d}. \tag{147}$$

In contrast, the energy density outside of the plates is given by a continuous set of Fourier modes. We note that the energy density *between* the plates is therefore *lower* than outside the plates. Thus, we expect a *force* on the plates. This *attractive force* is exerted by the vacuum. It is the Casimir force.

The expression for the ground state energy density is divergent. To calculate the Casimir force, we need to regularize the divergence. We do so by multiplying each mode by $e^{-\alpha\frac{n\pi}{d}}$ and take the limit $\alpha \to 0$ at the end of the calculation.
Now comes the key physical point about our treatment of (most) divergences in QFT: We need to distinguish divergences in unphysical (i.e., not measurable) quantities from divergences in physical (i.e., measurable quantities). Divergences in unphysical quantities are not necessarily a problem. Divergences in physical quantities are a problem and signal that the theory is not valid for the problem that we have applied it to.
What typically happens in calculations of loop effects (or in other words, calculations that involve quantum fluctuations) in QFT is, that unphysical quantities at some intermediate point of our calculation diverge. Measurable quantities stay finite. In some sense, they just mean that we have not set up our formalism in such a way that is best adapted to the physics.
In the concrete example, we clearly see that the ground-state energy density diverges. However, it is not itself observable. Instead, the Casimir force is observable, so the key question is whether or not the Casimir force diverges, when we remove the regularization.
Then, $F_{\text{Casimir}} = -\frac{\partial d(\Delta\epsilon)}{\partial d}$, where $\Delta\epsilon$ is the difference in energy densities and $d(\Delta\epsilon)$ the difference in energies.

$$\Delta\epsilon = \underbrace{\frac{\pi}{2d^2}\sum_{n=1}^{\infty} n}_{\substack{\text{energy density}\\ \text{between the plates}}} \underbrace{- \lim_{d\to\infty}\frac{\pi}{2d^2}\sum_{n=1}^{\infty} n}_{\substack{\text{energy density}\\ \text{outside the plates}}}. \tag{148}$$

We write the energy density outside the plates as $\lim_{d\to\infty}$, so that we can calculate the Casimir force by subtracting two regularized quantities from each other, because the difference of two divergent quantities is ill-defined.
Now we need to regularize the sum $\sum_{n=1}^{\infty} n$, and obtain:

$$F_{\text{Casimir}} = -\frac{\partial}{\partial d} d \lim_{\alpha\to 0}\left(\frac{\pi}{2d^2}\sum_{n=1}^{\infty} n e^{-\frac{\alpha n\pi}{d}} - \lim_{d\to\infty}\frac{\pi}{2d^2}\sum_{n=1}^{\infty} n e^{-\frac{\alpha n\pi}{d}}\right). \tag{149}$$

We use

$$\frac{\pi}{2d}\sum_{n=1}^{\infty} n e^{-\frac{\alpha n\pi}{d}} = -\frac{1}{2}\frac{\partial}{\partial\alpha}\sum_{n=1}^{\infty} e^{-\frac{\alpha n\pi}{d}} \tag{150}$$

$$= -\frac{1}{2}\frac{\partial}{\partial\alpha}\frac{1}{1-e^{-\alpha\pi/d}} \tag{151}$$

$$= \frac{1}{2}\frac{1}{(1-e^{-\alpha\pi/d})^2} e^{-\frac{\alpha\pi}{d}}\frac{\pi}{d}. \tag{152}$$

Thus,

$$\implies F_{\text{Casimir}} = -\frac{\partial}{\partial d} d \lim_{\alpha\to 0}\left(\frac{\pi}{2d^2}\frac{e^{\frac{\alpha\pi}{d}}}{(e^{\alpha\pi/d}-1)^2} - \lim_{d\to\infty}\frac{\pi}{2d^2}\frac{e^{\frac{\alpha\pi}{d}}}{(e^{\alpha\pi/d}-1)^2}\right). \tag{153}$$

Now we take the limit $\alpha \to 0$, for which we can expand the above expression up to the α^0 term. The series is a Laurent series, i.e., it starts with a negative power of α. This term diverges in the limit $\alpha \to 0$. This divergence is *not* an observable (measurable) quantity. In the measurable quantity, namely the force, this divergence cancels. As emphasized above, this is a first example of an important point in UV divergences in QFT, namely to carefully distinguish between divergences in physical and unphysical quantities.

In fact, we find that the divergent parts of the energy density between and outside the plates cancel:

$$F_{\text{Casimir}} = -\frac{\partial}{\partial d} d \lim_{\alpha\to 0}\left(\frac{1}{2\pi\alpha^2} - \frac{\pi}{24d^2} - \lim_{d\to\infty}\left(\frac{1}{2\pi\alpha^2} - \frac{\pi}{24d^2}\right)\right) \tag{154}$$

$$= -\frac{\partial}{\partial d}\left(-\frac{\pi}{24}\frac{d}{d^2}\right) = -\frac{\pi}{24d^2} \tag{155}$$

Thus, we are finding a finite, attractive force between the plates. The dependence on d has been tested experimentally, confirming our procedure. Note that the key point of properly dealing with the divergences was to focus on an *observable* quantity. In this quantity, two formally divergent expressions cancel, which makes the calculation subtle, but there is nothing problematic about it. Note that our result says that $\sum_{n=1}^{\infty} = -\frac{1}{12}$, at least in the context that we investigated here. In fact, this equality also shows up when doing ζ-function regularization and renormalization. There, the Casimir force is defined from an analytical continuation of the ζ-function, which implies the above result. In this method of dealing with divergent, intermediate results, the regularization and renormalization is done implicitly and it is conceptually less clear what one is doing.
Let us also stress that the final result for the Casimir force is *independent* of the choice of regularization; confirming that we are extracting a prediction for a physical quantity (which must not depend on arbitrarily chosen regularizations).

3.4 Quantum statistics

We know that multi-particle states which are antisymmetric (symmetric) under exchange of any two particles contain fermions (bosons). Which one do we have in our Fock space? Because $\left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\right] = 0$, we have that $|p_1 p_2\rangle = |p_2 p_1\rangle$ and similarly for states with more particles. In detail,

we see the link between the statistics and the choice of commutator (over anticommutator, for which $\{a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\} = a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} + a^\dagger_{\vec{q}} a^\dagger_{\vec{p}}$ as follows:

$$|p_1 p_2\rangle \quad = \quad a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |0\rangle = \left([a^\dagger_{\vec{p}_1}, a^\dagger_{\vec{p}_2}] + a^\dagger_{\vec{p}_2} a^\dagger_{\vec{p}_1} \right) |0\rangle = |p_2 p_1\rangle. \tag{156}$$

This means that the particles are *bosons*. This result is the first example of the spin-statistics theorem, which says that particles with integer spin are bosons and particles with half-integer spin are fermions. Mathematically, the correct quantization for bosons is the one using the commutator of creation and annihilation operators.
If we perform the same calculation with an anticommutator, we obtain $|p_2 p_1\rangle = -|p_1 p_2\rangle$, i.e., the state changes by a sign under the exchange of two particles. This is Fermi-Dirac statistics, and results in the Pauli exclusion principle, where the state $|p\, p\rangle = 0$, i.e., no two particles can be in the same state.
In the exercises, we will learn what goes wrong, if we try to quantize using the anticommutator (which would imply fermionic statistics, i.e., the Pauli exclusion principle). When we quantize spin 1/2 particles later in the lecture, we will learn that we must use the anticommutator, and that therefore they satisfy Fermi statistics and the Pauli exclusion principle.

3.5 Interpretation of $\phi(\vec{x})$

Now that we have constructed the Fock space, we can understand how $\phi(\vec{x})$ acts on the vacuum:

$$\phi(\vec{x})\,|0\rangle = ? \tag{157}$$

Mini-Exercise 9. Calculate the rhs of this expression. Use

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\, \tilde{\phi}(\vec{p}) \quad \text{and} \quad \tilde{\phi}(\vec{p}) = \frac{1}{\sqrt{2\omega_{\vec{p}}}}(a_{\vec{p}} + a^\dagger_{-\vec{p}}). \tag{158}$$

Solution.

$$\begin{aligned}
\phi(\vec{x})\,|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\, \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}} (a_{\vec{p}} + a^\dagger_{-\vec{p}})\,|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\, \frac{e^{i\vec{p}\cdot\vec{x}}}{\sqrt{2\omega_{\vec{p}}}} a^\dagger_{-\vec{p}}\,|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{\sqrt{2\omega_{\vec{p}}}} e^{-i\vec{p}\cdot\vec{x}}\, \underbrace{a^\dagger_{\vec{p}}\,|0\rangle}_{|\vec{p}\rangle} \\
&= \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2\omega_{\vec{p}}} e^{-i\vec{p}\cdot\vec{x}}\,|p\rangle\,.
\end{aligned}$$

This has the interpretation of a superposition of 1-particle states with different momenta. Because we are integrating over all momenta, the result only depends on $\vec{x}$, in other words, $\phi(\vec{x})\,|0\rangle$ is a one-particle state at position $\vec{x}$. We will use this information later, when we discuss *causality*.

3.6 Heisenberg picture

So far, we have worked in the Schrödinger picture, where the field operators $\phi(\vec{x})$ and $\Pi(\vec{x})$ have no time-dependence. Instead, the states evolve in time, $|p\rangle = |p(t=0)\rangle$ and $|p(t)\rangle = e^{-iHt}\,|p(t=0)\rangle$. The relations between field operators and states that we have used so far are relations at $t = 0$.
It is clearly not natural in a Poincaré-invariant theory to separate temporal and spatial dependence from each other. Instead, we switch to the *Heisenberg picture*, where operators depend on space *and* time and states are held fixed. The time dependence of any operator $\mathcal{O}$ follows from the requirement

$$\underbrace{\langle\psi|\mathcal{O}(t)|\psi\rangle}_{\text{Heisenberg}} = \underbrace{\langle\psi(t)|\mathcal{O}|\psi(t)\rangle}_{\text{Schrödinger}}. \tag{159}$$

In particular, the field operator becomes

$$\phi(x) = e^{iHt}\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\Big(a_{\vec{p}}\,e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\,e^{-i\vec{p}\cdot\vec{x}}\Big)e^{-iHt} \tag{160}$$

$$= \underbrace{\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}}_{(*)}\Big(a_{\vec{p}}\,e^{-ipx} + a^\dagger_{\vec{p}}\,e^{ipx}\Big) \tag{161}$$

For the Lorentz invariance of the measure $(*)$, see our discussion of state normalization. Note that we now have 4-vector products in the exponents, i.e., $px = p_\mu x^\mu$, which are also Lorentz invariant.

3.7 Causality

Causality is a key property of a QFT. Causality implies that operators at spacelike distances commute and can therefore be measured simultaneously. It is one of the properties that is not automatically built into Quantum Mechanics and that motivated us to develop a new formalism, the QFT formalism.
In particular, the requirement that operators at spacelike distance should commute holds for $\phi(x)$, i.e., we need $[\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$ for causality to hold. Physically, this means that the creation of a particle at x cannot affect the creation of a particle at y, if x and y are spacelike separated.
Note that the equal-time-commutation relations in the Schrödinger picture, $[\phi(\vec{x}),\phi(\vec{y})] = 0$, do not immediately imply causality; we need to check the commutator in the Heisenberg picture to have the full spacetime-dependence of the operator.

$$[\phi(x),\phi(y)] = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{p}}}}\int \frac{d^3q}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{q}}}}\Big(\Big[a_{\vec{p}},a^\dagger_{\vec{q}}\Big]e^{-ipx+iqy} + \Big[a^\dagger_{\vec{p}},a_{\vec{q}}\Big]e^{ipx-iqy}\Big), \tag{162}$$

where we directly set the vanishing commutators among two a's or two $a^\dagger$'s to zero.

$$[\phi(x),\phi(y)] = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}e^{-ip(x-y)} - \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}e^{ip(x-y)}. \tag{163}$$

This expression does *not* vanish for timelike distances, e.g., taking $(x-y) = (t,0,0,0)$, we get

$$[\phi(\vec{x},0),\phi(\vec{x},t)] \sim e^{-i\omega_{\vec{p}}t} - e^{i\omega_{\vec{p}}t}. \tag{164}$$

This expression vanishes for spacelike separation. This follows, because $[\phi(\vec{x},t),\phi(\vec{y},t)]$ vanishes, which we can see from

$$[\phi(\vec{x},t),\phi(\vec{y},t)] = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{\omega_{\vec{p}}}\Big(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}\Big) \tag{165}$$

$$= 0. \tag{166}$$

We have flipped the sign of $\vec{p}$ by changing the integration variable $\vec{p} \to -\vec{p}$ in the second term. However, $[\phi(x),\phi(y)]$ must be a Lorentz invariant expression, because it is based on the Lorentz invariant integration measure. Thus, it can only depend on $(x-y)^2$ and must therefore vanish for all $(x-y)^2 < 0$, irrespective of whether the two times are equal, since a boost of a spacelike interval can always be used to bring the two points to equal times.
Thus, causality is respected by our theory - which is not surprising, because we're basing it on Lorentz invariance and so the notion that nothing can propagate faster then light is built in.

3.8 A note on quantum entanglement:

Locality in QFT implies that local operators, defined at a single spacetime point, commute at spacelike distances. At a first glance, one might wonder whether locality in QFT is not in contradiction to entanglement in QM, where states can be entangled over spacelike distances, e.g., in EPR-states. However, the fact that in QFT local operators commute is *not* at odds with the existence of entangled states. Consider the following example in QM, an entangled state of two spin 1/2 particles, which can be at large spatial distance

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle). \tag{167}$$

S_1, measuring the spin or particle 1, and S_2, measuring the spin of particle 2, commute nevertheless, and the expectation value of S_2 is not changed by measuring S_1.
Just like the states in QM can be non-local, the states in QFT are generically non-local; in our discussion of the path-integral formalism we will see that states are represented as functionals of the field configuration and depend on non-local information.

3.9 Propagators and causality

Literature suggestion: Peskin/Schröder.

There is a different question we could ask to probe the causal structure of the theory. We prepare a particle at spacetime point y, by acting with $\phi(y)$ on the vacuum. We the ask that the probability amplitude is to find the particle at $\phi(x)$. This is encoded in the *propagator*

$$D(x-y) = \underbrace{\langle 0|\,\phi(x)}_{\text{particle at x}} \quad \underbrace{\phi(y)\,|0\rangle}_{\text{particle at y}} = \langle 0|\,\phi(x)\phi(y)\,|0\rangle\,, \tag{168}$$

which is the probability amplitude for the particle to *propagate* from y to x.
We study its structure here for two reasons: First, at a physical level, we will achieve a crucial insight into the physical reason why antiparticles (particles with the same mass and spin, but opposite charges under internal symmetries, e.g., electric charge) must exist. Second, the techniques

of complex analysis that we will use, are useful in many other contexts in QFT and beyond, and the propagator is a useful example to practice them with.
We might first expect that the propagator should vanish for spacelike distances. To evaluate $D(x-y)$, we write

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \frac{1}{\sqrt{2\omega_{\vec{q}}}} \langle 0| \left(a_{\vec{q}}\, e^{-iqx} + a^\dagger_{\vec{q}}\, e^{iqx}\right) \cdot \left(a_{\vec{p}}\, e^{-ipy} + a^\dagger_{\vec{p}}\, e^{ipy}\right) |0\rangle \tag{169}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)}, \qquad \text{using } a_{\vec{p}}\,|0\rangle = 0 = \langle 0|\, a^\dagger_{\vec{q}} \quad \text{and } [a_{\vec{p}}, a^\dagger_{\vec{q}}] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}). \tag{170}$$

This expression already looks like it will be non-zero for spacelike distances, but let us evaluate it to see what exactly its form is.
First, we reduce the three-dimensional integral to a one-dimensional integral over the modulus of $\vec{p}$, i.e., $|\vec{p}|$. This works as follows: Because we focus on spacelike distances ($(x-y)^2 < 0$), we can pick $x^0 = y^0$ and name $\vec{x} - \vec{y} = \vec{r}$, so that

$$D(x-y)\Big|_{(x-y)^2<0} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}} \tag{171}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty d|\vec{p}|\, \frac{|\vec{p}|^2}{2\omega_{\vec{p}}} \frac{e^{i|\vec{p}||\vec{r}|} - e^{-i|\vec{p}||\vec{r}|}}{i|\vec{p}||\vec{r}|}. \tag{172}$$

In this step, we rewrote

$$\int d^3p = \int_0^\infty d|\vec{p}|\, |\vec{p}|^2 \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi \quad \text{and} \quad \vec{p}\cdot\vec{r} = |\vec{p}||\vec{r}|\cos\theta. \tag{173}$$

We used

$$\int_0^\pi d\theta\, \sin\theta e^{i\alpha\cos\theta} = \frac{2\sin\alpha}{\alpha} \quad \text{and} \quad 2\sin\alpha = ie^{-i\alpha} - ie^{i\alpha} \quad \text{for } \alpha = |\vec{p}||\vec{r}|. \tag{174}$$

In the next step, we change variables according to $|\vec{p}| \to -|\vec{p}|$ in the 2nd term, so that the integral, instead of ranging from 0 to ∞, ranges from $-\infty$ to $+\infty$.
We thus have that

$$D(x-y)\Big|_{(x-y)^2<0} = \frac{-i}{2(2\pi)^2} |\vec{r}|^{-1} \int_{-\infty}^\infty d|\vec{p}|\, \frac{|\vec{p}| e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2 + m^2}}. \tag{175}$$

This is actually an integral representation of a Hankel function, so we could just use that result, together with the asymptotic form of the particular Hankel function for large r to obtain $D(x-y)\big|_{(x-y)^2<0} \sim e^{-m\,r}$. However, to get a better idea why that happens, and to practice techniques of complex analysis, we will arrive at that result differently.
In the complex plane (in this case, the complex-$|\vec{p}|$-plane), we can use Cauchy's integral theorem. The theorem says that the integral of a function over a closed curve in the complex plane vanishes, if the function is infinitely differentiable and locally identical to its Fourier series (i.e., if it is a holomorphic function). (If the function has poles, then the integral will pick up the corresponding residue.) This allows us to *deform* integration contours, by starting from the integral along the

real axis and completing it by some contour through the complex plane, so that together, the real axis and the new part of the contour form a closed curve. By the theorem, the original integral along the real axis is equal to (minus) the integral along that contour.
What we have to watch out for when doing such deformations, are poles (where the function is singular) and branch cuts (where the limit of the function, taken from both sides of the branch cut, is not equal, i.e., the function is multi-valued).
We encounter a simple example of a branch cut for the function $\sqrt{z}$ with $z \in \mathbb{C}$. This example will be relevant for our case. In the complex plane, we can parameterize z through its modulus and a phase

$$\sqrt{z} = \sqrt{r\, e^{i\theta}} = \sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)\right), \text{ with } \theta \in (-\pi, \pi). \tag{176}$$

For $z \in \mathbb{R}$, we have that $\theta = 0$. Positive imaginary z has $\theta = \pi/2$ and negative imaginary z has $\theta = -\pi/2$. Negative real z can be approached coming from positive imaginary parts (with $\theta \to \pi$) or from negative imaginary parts (with $\theta \to -\pi$). It turns out that $\sqrt{z}$ is discontinuous across the negative real axis, i.e., it has a branch cut. We can see the discontinuity as follows:

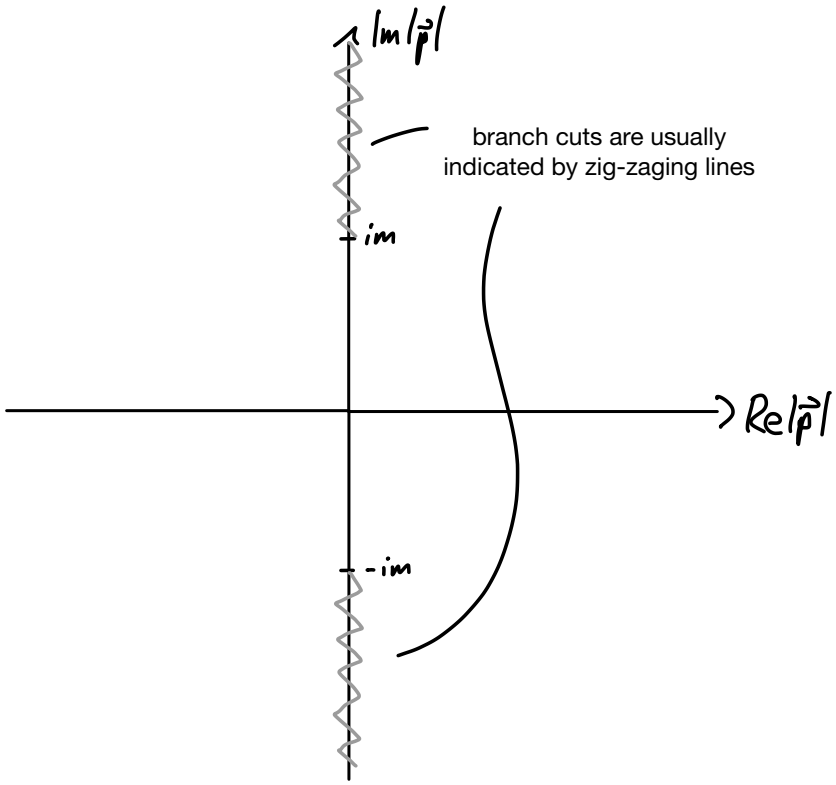
$$\lim_{(r,\theta)\to(r_0,\pi)} \sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)\right) = \sqrt{r_0}\, i, \tag{177}$$

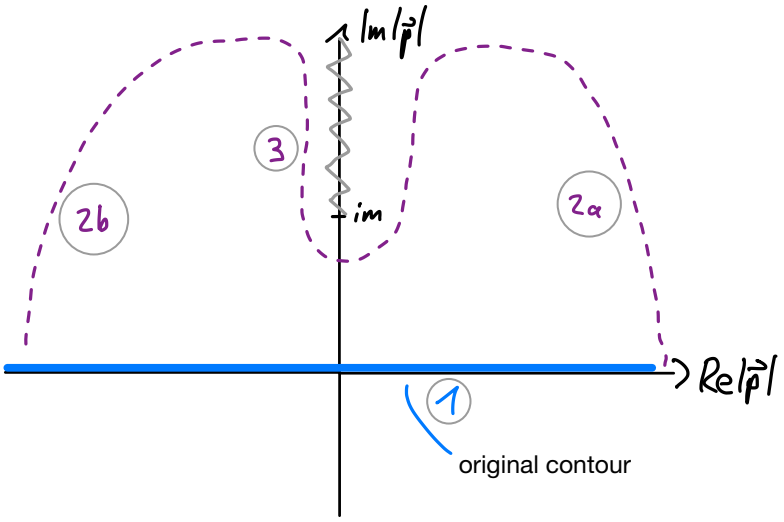
$$\lim_{(r,\theta)\to(r_0,-\pi)} \sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)\right) = \sqrt{r_0}\,(-i). \tag{178}$$

In the integral that we are interested, there is a branch cut when $|\vec{p}|^2 + m^2$ becomes negative. This translates into $|\vec{p}|$ being purely imaginary and the imaginary part either being positive and greater than m, or negative and smaller than $-m$.
Therefore, there are branch cuts on the imaginary axis, starting at $\pm im$. This means that the integrand is discontinuous across this line:

$$\lim_{\varepsilon\to 0} \frac{|\vec{p}|e^{i|\vec{p}|r}}{\sqrt{|\vec{p}|^2 + m^2}}\bigg|_{|\vec{p}|=im+\varepsilon} \neq \lim_{\varepsilon\to 0} \frac{|\vec{p}|e^{i|\vec{p}|r}}{\sqrt{|\vec{p}|^2 + m^2}}\bigg|_{|\vec{p}|=im-\varepsilon} \tag{179}$$

The branch cuts start in the singular branch points at $|\vec{p}| = \pm im$. In the presence of a branch cut, we shift the integration contour upwards, so that it wraps around the branch cut, but never crosses it. The difference between the original contour and this contour vanishes because of Cauchy's theorem: Due to Cauchy's residue theorem, the integral along the curve $C = \textcircled{1} + \textcircled{2a} + \textcircled{3} + \textcircled{2b}$ vanishes, because there's no singularity that is enclosed. In addition, the integrals along $\textcircled{2a}$ and $\textcircled{2b}$ (which are meant to lie at infinity) vanish. Thus, the integral along $\textcircled{1}$ is equal to the integral along $\textcircled{3}$ (followed from right to left).
Because we saw previously that $\sqrt{z}$ differs by an overall sign across the branch cut, the integral





along the right and the left part of the branch cut add up and we obtain

$$
\begin{aligned}
D(x-y)\Big|_{(x-y)^2<0} &= \frac{-i}{2(2\pi)^2}|\vec{r}|\lim_{\epsilon\to 0}\left(\int_{i\infty+\epsilon}^{im+\epsilon} d|\vec{p}|\,\frac{|\vec{p}|e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2+m^2}}+\int_{im-\epsilon}^{i\infty-\epsilon} d|\vec{p}|\,\frac{|\vec{p}|e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2+m^2}}\right)\\
&= \frac{-i}{(2\pi)^2}|\vec{r}|\int_{im}^{i\infty} d|\vec{p}|\frac{|\vec{p}|e^{i|\vec{p}||\vec{r}|}}{\sqrt{|\vec{p}|^2+m^2}}\\
&= \frac{1}{(2\pi)^2}\frac{1}{|\vec{r}|}\int_m^\infty d\rho\,\rho\,\frac{e^{-\rho|\vec{r}|}}{\sqrt{\rho^2-m^2}},
\end{aligned}
\tag{180}
$$

where in the last step we defined $\rho = -i|\vec{p}|$.
We are only interested in the behavior at $rm \gg 1$, i.e., distances large compared to the scale set by the mass, in which case the integrand is suppressed for all values of ρ, except for the point $\rho = m$. Thus we obtain

$$D(x-y)\Big|_{(x-y)^2<0} \overset{|\vec{r}|m\gg 1}{=} e^{-m|\vec{r}|}. \tag{181}$$

Thus, there is a nonzero probability amplitude for a particle to propagate outside the lightcone. This seems worrisome. Does this mean that our theory violates causality? The answer is no, because this does not affect *measurements*, i.e., it is a property of the theory we can never test in an experiment (or use for faster-than-light signalling). In fact,

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0 \qquad \text{for } (x-y)^2 < 0. \tag{182}$$

Thus, the quantum amplitude for the particle to propagate over a spacelike distance from y to x *interferes destructively* with the quantum amplitude for a particle to propagate from x to y. Such a destructive interference makes sense, because, if x and y are spacelike to each other, the temporal order of the two is not fixed and can be changed. Therefore, it is equally viable to consider propagation from x to y as it is from y to x and thus both processes occur. Their quantum amplitudes cancel when we consider a measurement.
Note that this would be different, if x and y were timelike to each other, because then only one direction of travel makes sense; the other one would be against the direction of time.
In a complex scalar theory, it gets even more interesting, because we can consider the commutator

$$\left[\phi(x), \phi^\dagger(y)\right] = \underbrace{\langle 0|\, \phi(x)\, \phi^\dagger(y)\, |0\rangle}_{\substack{\text{describes particle} \\ \text{propagating from} \\ y \text{ to } x}} - \underbrace{\langle 0|\, \phi^\dagger(y)\, \phi(x)\, |0\rangle}_{\substack{\text{describes } \textit{antiparticle} \\ \text{propagating from} \\ x \text{ to } y}}. \tag{183}$$

This is a profound result: it tells us that antiparticles *must* exist for QFT to be causal. It is *not* a choice to have or not have antiparticles. They are a *necessity* for a causal theory. (In the case of a real scalar field we did not see that clearly, because an uncharged scalar, described by a real scalar field, is its own antiparticle.)

4 Path-integral quantization for the scalar field

(See Gelis, Ryder, Franklin, Srednicki)
This approach to quantum physics places the superposition principle - that more than one classical realization of a system is realized at the same time - center stage. For a single particle, it means that if it is in a momentum eigenstates, it is in a superposition of position eigenstates, e.g., completely delocalized. For a field, it means that the relevant entity is *not* a single configuration of the scalar field (which we can associate to particles at some positions). Rather, all field configurations which are compatible with boundary of initial conditions are realized at the same time and interfere destructively or constructively. In terms of particles, this means that we never have a constant number of particles, but that in addition to the real particles, there are *virtual* particles in our theory, which are only there for short amounts of time, before they disappear again.

Mathematically, instead of working with operators on a Hilbert space, we work with functionals, i.e., maps from the space of functions (field configurations) to the real (complex) numbers. We will introduce the necessary mathematical concepts as we go along and partially in the exercises.

Conceptually, besides providing a different (complementary) intuition about QFT than the canonical formalism, the path-integral formalism is also manifestly Lorentz invariant.

In Quantum Mechanics and in Quantum Field Theory, the canonical formalism and the path-integral formalism are equivalent. In quantum gravity, the situation is unclear. There are approaches to quantum gravity, in which spacetime as a whole is quantized in such a way that a Hamiltonian cannot be written down, just an action and a path integral (e.g., causal set theory).

Our plan for the next few lectures is to:

- derive path integral for QM
- generalize path integral to QFT
- introduce interactions
- first application: introduce interactions and understand the effect of quantum fluctuations: symmetry breaking in the one-loop effective potential.

4.1 Path-integral for Quantum Mechanics

We will now derive an expression for the probability amplitude for a particle to propagate from position q_i to position q_f that is a superposition of all paths, each one weighted with a complex amplitude that results in (constructive or destructive) interference between paths.
Consider QM for a single classical degree of freedom, described by the Hamiltonian

$$H = \frac{P^2}{2m} + V(Q), \tag{184}$$

with momentum operator P and position operator Q, and commutator $[Q, P] = i$. The probability for the particle to start at the initial position q_i and end at the final position q_f after time $t_f - t_i$

is given by:

$$\left|\langle q_f|\, e^{-iH(t_f-t_i)}\, |q_i\rangle\right|^2. \tag{185}$$

We will derive the path-integral expression for the amplitude. The intuition underlying the path integral can be obtained by starting from the double slit, in which this transition has two major contributions, one from each path, see left panel in Fig. 3. Then we imagine generalizing the 2 slits to n slits and the one barrier to m barriers, see right panel in Fig. 3.

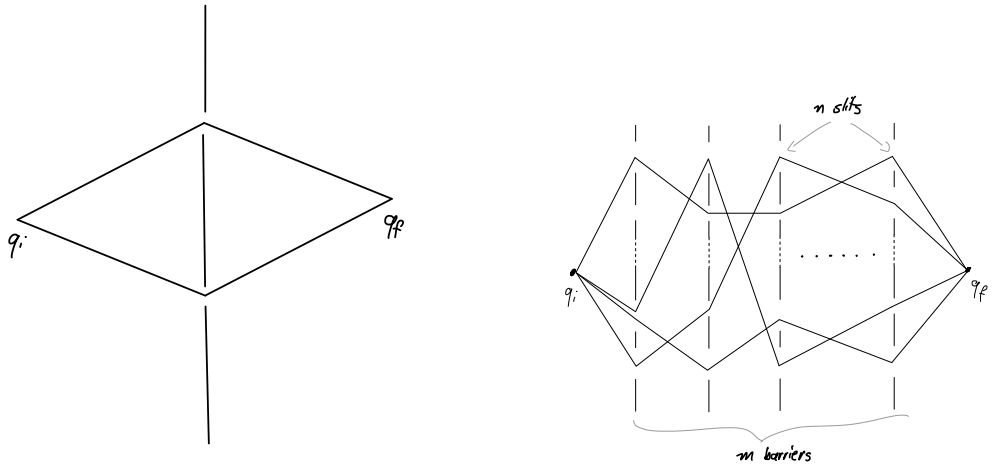


Figure 3: Left panel: we show the two possible paths in the double-slit experiment. Right panel: we show the generalization to m barriers with n slits in each and a subset of all possible paths in this setup.

As we take $n \to \infty$ and $m \to \infty$, we obtain all possible paths between q_i and q_f that a freely propagating particle can take, see Fig. 4.

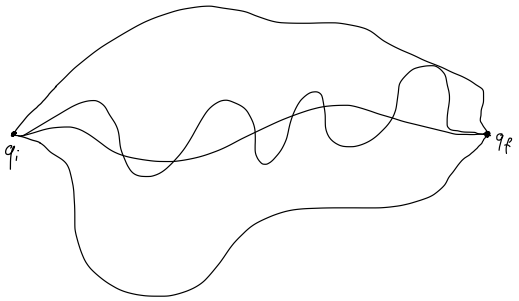


Figure 4: The generalization contains all possible paths that we can imagine, some of which are shown in this sketch to illustrate the concept.

This is what the propagating particle (in a potential $V(Q)$) does. Just like the two paths interfere in the double-slit setup, all paths for the freely propagating particle interfere. As it turns out, paths far away from the classical path interfere destructively, so that the transition probability amplitude is dominated by the classical path and fluctuations around it.

Before we derive the expression for the path integral and the expression for the quantum amplitude associated to each path, we can already develop some intuition for what the result should be. We

take the following steps in our argument that motivates the result:

1) We are looking for a quantum amplitude, i.e., a quantity that can in general be complex and whose absolute value can never exceed 1, so that it has a probability interpretation. This suggests that we can write the quantum amplitude as $\exp[i\,A]$, with some A that we need to determine and that is real and depends on the path that we are considering, i.e., $A = A[\text{path}]$.

2) We consider how constructive and destructive interference between different paths can be encoded in this quantity: if we compare the quantum amplitudes $\exp[i\,A[\text{path}]]$ for two different paths, and they differ exactly by their sign, then they cancel in the final expression. In contrast, if the amplitudes are close to each other in their value in the complex plane, then they contribute to the final expression.

3) We consider the classical path for the freely propagating particle. We know from Quantum Mechanics lectures that the expectation value of the position of the particle should follow the classical path. Therefore, we want that $\exp[i\,A[\text{path}]]$ varies slowly across different paths that are close to the classical one. This already gives us a hint, what $A[\text{path}]$ could be, because we know that the *action* has an extremum for the classical path and therefore varies slowly for paths close to it.

4) We cross-check our expectation that $\exp[i\,S[\text{path}]]$, with S the action, could be a good candidate for the quantum amplitude for each path with the expectation that A should be a Lorentz-invariant expression, which S indeed is.

This line of reasoning leads us to a *well-motivated expectation* that the path integral may contain a factor $e^{i\,S}$ for each path that will be included.
Of course this is absolutely *not* a derivation; it's just a way to obtain some intuition for what we might expect. We will now do a proper derivation of the path integral.
For the derivation, our ultimate goal is to start from the transition amplitude, written in terms of operators acting on states, and convert it into a (function) integral over all paths, where each path is weighted by a quantum amplitude that is a *complex number*, *not an operator*. Thus, the steps of our derivation will be aimed at getting rid of operators and exchanging them for eigenvalues.
We subdivide the time interval $[t_i, t_f]$ into N small time intervals, so that we can keep track of what the particle does in the small intervals. We introduce

$$\Delta = \frac{t_f - t_i}{N}, \qquad t_n = t_i + n\Delta, \tag{186}$$

so $t_0 = t_i$, $t_N = t_f$, so that we can write the transition amplitude in terms of the many intermediate positions.
Now we can factorize

$$e^{-iH(t_f - t_i)} = e^{-iH(t_N - t_{N-1})} e^{-iH(t_{N-1} - t_{N-2})} \dots e^{-iH(t_1 - t_0)}, \tag{187}$$

which is possible because H at successive times commute. Between these successive factors on the right-hand-side, we can insert the identity operator as a sum over position eigenstates, in order to

express the transition amplitude in terms of the positions at successive times:

$$\mathbb{1} = \int_{-\infty}^{\infty} dq \; |q\rangle\langle q| \,. \tag{188}$$

Because we will need the identity operator multiple times, and it would lead to confusion if each of the dummy integration variables was called q, we will instead introduce dummy variables q_j with $j = 1, 2, \ldots$ for the identity operator inserted at successive times t_1, t_2, ... This will eventually allows us to convert the $V(Q)$ in H into $V(q_n)$, the position at time t_n.

$$\langle q_f| \, e^{-iH(t_f - t_i)} \, |q_i\rangle = \int \prod_{j=1}^{N-1} dq_j \; \langle q_f| \, e^{-iH\Delta} \, |q_{N-1}\rangle\langle q_{N-1}| \, e^{-iH\Delta} \, |q_{N-2}\rangle \ldots \langle q_1| \, e^{-iH\Delta} \, |q_i\rangle . \tag{189}$$

Because the time interval consists of N parts, there are $N-1$ factors of $\mathbb{1}$ to insert inbetween successive factors of $e^{-iH\Delta}$. Now, there is a slight complication, namely that the two terms in H, $\frac{P^2}{2m}$ and $V(Q)$, do not commute, because $[Q, P] = i$. This makes it difficult to split $e^{iH\Delta}$ into $e^{i\frac{P^2}{2m}}$ and $e^{iV(Q)}$ and use $e^{iV(Q)} \, |q_n\rangle = e^{iV(q_n)} \, |q_n\rangle$.
However, we can use a trick: We can use the Baker-Campbell-Hausdorff-formula:

$$e^{\Delta(A+B)} = e^{\Delta A} e^{\Delta B} e^{-\frac{\Delta^2}{2}[A,B] + \mathcal{O}(\Delta^3)}. \tag{190}$$

For $\Delta \to 0$ (i.e., $N \to \infty$), the Δ^2 - and all higher-order factors are negligible. Then, in the limit $\Delta \to 0$, we can use

$$\langle q_{i+1}| \, e^{-i\Delta H} \, |q_i\rangle = \langle q_{i+1}| \, e^{-i\Delta \frac{P^2}{2m}} e^{-i\Delta V(Q)} \, |q_i\rangle + \mathcal{O}(\Delta^2) \tag{191}$$

$$= \langle q_{i+1}| \, e^{-i\Delta \frac{P^2}{2m}} e^{-i\Delta V(q_i)} \, |q_i\rangle + \mathcal{O}(\Delta^2). \tag{192}$$

This contains the eigenvalue q_i of the position operator, but it still contains the momentum operator P itself. To convert it into its eigenvalue, we insert the identity operator written in terms of momentum eigenstates

$$\int \frac{dp}{2\pi} \; |p\rangle\langle p| = \mathbb{1}. \tag{193}$$

Just like for the position operator, we will use a dummy integration variable labelled by the time at which we insert the identity into the whole expression. Thus we obtain

$$\langle q_{i+1}| \, e^{-i\Delta H} \, |q_i\rangle = \int \frac{dp_i}{2\pi} \; \langle q_{i+1}| \, e^{-i\Delta \frac{p_i}{2m}} \, |p_i\rangle \, e^{-i\Delta V(q_i)} \langle p_i|q_i\rangle + \mathcal{O}(\Delta^2). \tag{194}$$

We already exchanged the operators in our expression for their eigenvalues, thus we can also pull the factor $e^{-i\Delta \frac{p_i}{2m}}$ out of the matrix element. However, we still have states, which we want to exchange for (complex) numbers next. Thus, we use that

$$\langle q|p\rangle = e^{ipq}, \tag{195}$$

and arrive at

$$\langle q_{i+1}| \, e^{-i\Delta H} \, |q_i\rangle = \int \frac{dp_i}{2\pi} e^{-i\Delta H(p_i, q_i)} e^{ip_i(q_{i+1} - q_i)} + \mathcal{O}(\Delta^2). \tag{196}$$

Note that $H(p_i, q_i)$ is now a *number*, no longer an operator and the right-hand side in general no longer contains operators (just their eigenvalues), nor states.

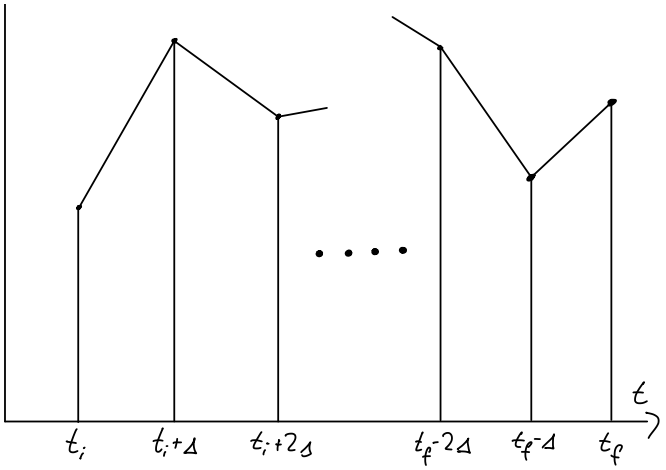


Figure 5: We illustrate the position of the particle at consecutive moments in time, as it enters the expression (197). There is no actual "path"; the connecting lines between the points are just to guide the eye. The "paths" are *not* continuously differentiable.

It remains to interpret $e^{-i\Delta H(p_i,q_i)}e^{ip_i(q_{i+1}-q_i)}$. To do so, we next note that $\frac{q_{i+1}-q_i}{\Delta} = \dot{q}_i$ is the discretization of the time-derivative of q_i. Overall, we thus have

$$\langle q_f|\, e^{-iH(t_f-t_i)}\, |q_i\rangle = \lim_{N\to\infty}\left(\int\prod_{j=1}^{N-1} dq_j\right)\left(\prod_{j=1}^{N}\frac{dp_j}{2\pi}\right)e^{-i\Delta H(p_j,q_j)}e^{i\Delta p_j\dot{q}_j}. \tag{197}$$

Let us illustrate this expression, see Fig. 5. Note that there are $N-1$ integrations over q_j, because we inserted a $\not\Vdash$ expressed in position eigenstates inbetween each of the factors of $e^{i\Delta H}$. This gives rise to N matrix elements of the form $\langle q_{i+1}|\, e^{-i\Delta H}\, |q_i\rangle$ and we insert a $\not\Vdash$ expressed in terms of momentum eigenstates inside each such matrix elements, so that we end up with N integrations over p_j.

Taking the $N\to\infty$ limit, we see that $e^{-i\,\Delta\, H(p_i,q_i)}$ will become $e^{-i\int_{t_i}^{t_f} dt\, H(p(t),q(t))}$ (and similarly for the other factor in the exponential. We thus obtain

$$\boxed{\langle q_f|\, e^{-iH(t_f-t_i)}\, |q_i\rangle = \mathcal{N}\int_{\substack{q(t_i)=q_i\\ q(t_f)=q_f}}\mathcal{D}p(t)\mathcal{D}q(t)\, e^{i\int_{t_i}^{t_f} dt\,(p(t)\dot{q}(t)-H(p,q))},} \tag{198}$$

where we introduced a normalization $\mathcal{N}$. This is the path-integral representation of the transition amplitude in its phase-space form. The measures $\mathcal{D}p(t)$ and $\mathcal{D}q(t)$ indicate that we are *not* integrating over *numbers* dp, dq, but over *functions* $q(t)$ and $p(t)$. The path-integral is therefore a *functional* integral.

Comments:

- There are no initial and final conditions on the momentum, because the position is sharp at beginning and end, and so the momentum must be totally unconstrained.
- The right-hand-side contains ordinary commuting numbers, while the light-hand-side contains operators.
- We *derived* the path integral from the canonical formulation of QM. However, we may also

"forget" about this origin and view the path integral as the *definition* of the quantum theory.

- The physical intuition behind the path integral is that all possible phase-space configurations $\{p(t), q(t)\}$ are realized simultaneously and each is weighted by a *phase* factor $e^{i\int dt\,(p\dot{q}-H)}$, which encodes quantum mechanical interference.
- The functional measure $\mathcal{D}q(t)\mathcal{D}p(t)$ in general lacks a solid mathematical foundation. Nevertheless, the formalism allows us to make progress in QFT, where the canonical formalism would prove extremely challenging and cumbersome. Wherever both formalisms can be used in practical calculations, they produce results that are in agreement.

In QFT, the form of the path integral that is used most is not a phase-space path integral, but one, where the momentum-integration has already been performed.
This can be done in theories in which the momentum only occurs quadratically, as we have assumed. In this case, we can perform the integral over $p(t)$, because it is a *Gaussian functional integral*.
We generalize

$$\int_{-\infty}^{\infty} dx\, e^{\pm\frac{ix^2}{2\sigma}} = \sqrt{2\pi\sigma}e^{\pm i\frac{\pi}{4}}. \tag{199}$$

(Note that this looks as if we took $\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2\sigma}} = \sqrt{2\pi\sigma}$ and simply made it complex. This is not the case. The correct derivation relies on the integral along the real axis, but then uses Cauchy's residue theorem.)

To use this, we go back to the path integral *before* taking the $N \to \infty$ limit. We treat each p_i as an independent variable and perform the Gaussian integral to obtain

$$\int \mathcal{D}p\, e^{i\Delta(p\dot{q}-\frac{p^2}{2m})} = \underbrace{e^{i\frac{\pi}{4}}\sqrt{\frac{2\pi m}{\Delta}}}_{\substack{\text{this prefactor is}\\ \text{independent of } q,\dot{q}}} e^{i\frac{\Delta m\dot{q}^2}{2}}. \tag{200}$$

We absorb the prefactor into the (undetermined) normalization constant $\mathcal{N}$. We will soon see that $\mathcal{N}$ drops out of computing *observables*. Therefore, it does not matter that $\mathcal{N}$ contains a factor that diverges for $\Delta \to 0$. We thus arrive at the path integral in the form

$$\langle q_f|\, e^{-iH(t_f-t_i)}\, |q_i\rangle = \int_{\substack{q(t_i)=q_i\\ q(t_f)=q_f}} \mathcal{D}q(t)\, e^{i\int_{t_i}^{t_f} dt\, \mathcal{L}(q(t))}, \tag{201}$$

because $L = \frac{m\dot{q}^2}{2} - V(q)$. Therefore,

$$\langle q_f|\, e^{-iH(t_f-t_i)}\, |q_i\rangle = \int_{\substack{q(t_i)=q_i\\ q(t_f)=q_f}} \mathcal{D}q(t)\, e^{iS[q(t)]}. \tag{202}$$

This is also known as the Feynman-Kac-formula.

4.2 Classical action, least-action principle

We have worked with $\hbar = 1$, so let us briefly reinstate $\hbar$. We know that $[S] = [\hbar]$ and we know that the argument of the exp cannot contain units. Thus, without redoing the calculation with $\hbar$

back in place, we know that the result has to be

$$\langle q_f|\, e^{-i\frac{H}{\hbar}(t_f-t_i)}\,|q_i\rangle = \int_{\substack{q(t_i)=q_i\\ q(t_f)=q_f}} \mathcal{D}q\; e^{\frac{i}{\hbar}S[q]}. \tag{203}$$

The rhs is a wildly oscillatory integral, with the following properties:

1) $e^{\frac{i}{\hbar}S}$ is a pure *phase factor*, i.e., $\left|e^{\frac{i}{\hbar}S}\right| = 1$ for all $q(t)$.

2) If $S[q(t)]$ changes slowly over neighboring paths, these contribute with a similar phase to the overall integral, i.e., they interfere *constructively*.

3) If $S[q(t)]$ changes rapidly across neighboring paths, we find that the contributions *cancel* each other, because they, roughly speaking, contribute with opposite sign.

Mini-Exercise 10. Based on properties 2) and 3), argue why you would expect a main contribution to the path integral from the classical paths, i.e., those paths that satisfy the classical equations of motion, and paths close to them.

Solution. Classical paths satisfy $\delta S = 0$, i.e., the action varies slowly.

4.3 Time-ordered products and generating functional

Before we make the transition to QFT, we need to develop a few more tools. With the QFT-application in mind, we will in particular develop those tools useful in QFT, which may not be the most important quantities in QM. Thus, why we are focusing on the specific quantities that we are will become clearer later in the course. For now we are just developing a toolbox, and we will come back to asking physical questions soon. When we will do so, we will have the tools available to answer them.

We are not only interested in transition amplitudes, but also in other quantities, e.g., expectation values. We will now see that these also have a path-integral representation. Consider the matrix element

$$\langle q_f|\, e^{-iH(t_f-t_I)}Qe^{-iH(t_I-t_i)}\,|q_i\rangle\,, \tag{204}$$

which measures the amplitude for the transition between q_i and q_f with Q acting at the intermediate time t_I, with $t_i < t_I < t_f$. To evaluate this, we write

$$Q = \int dq\, dq'\;\; |q\rangle \underbrace{\langle q|\, Q\,|q'\rangle}_{q\,\delta(q-q')} \langle q'| \tag{205}$$

$$= \int dq\; q\,|q\rangle\langle q|\,. \tag{206}$$

By repeating the remainder of our previous path-integral derivation, but with this insertion, we arrive at

$$\langle q_f|\, e^{-iH(t_f-t_I)}Qe^{-iH(t_I-t_i)}\,|q_i\rangle = \int_{\substack{q(t_i)=q_i\\ q(t_f)=q_f}} \mathcal{D}q\; q(t_I)e^{iS[q]}. \tag{207}$$

Similarly, we can consider several $Q's$ inserted at different times. In QM, these are not necessarily the quantities we are most interested in. However, as we will see later in the course, when we discuss

the scattering of particles, the analogous expressions in QFT are central to the theory. Thus, we check that they can be written in terms of the path integral in QM. If we have $Q(t_1)Q(t_2)$, with $Q(t) = e^{iHt}Qe^{-iHt}$, we see that the order of the $Q's$ matters, because Q does not commute with the P^2-term inside H. Thus we consider the time-ordered products

$$TQ(t_1)Q(t_2) = \begin{cases} Q(t_1)Q(t_2), & \text{if } t_1 \geqslant t_2 \\ Q(t_2)Q(t_1), & \text{if } t_2 > t_1 \end{cases} \tag{208}$$

(with $Q(t) = e^{iHt}Qe^{-iHt}$ and $|q,t\rangle = e^{iHt}\,|q\rangle$.)
We have, by the same procedure of inserting $Q(t_2/t_1)$ in the derivation,

$$\langle q_f, t_f|\, T(Q(t_1)Q(t_2))\, |q_i, t_i\rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q\; q(t_1)q(t_2)e^{iS[q]}. \tag{209}$$

Note that the right-hand-side contains *commuting numbers* $q(t_1)$, $q(t_2)$, thus the time-ordering on the left is crucial for the equality, because otherwise we would have to account for extra commutators that have no counterpart on the right-hand-side. The generalization to n factors is direct

$$\langle q_f, t_f|\, T(Q(t_1)\dots Q(t_n))\, |q_i, t_i\rangle = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q\; q(t_1)\dots q(t_2)e^{iS[q]}. \tag{210}$$

In QM, these are not necessarily the matrix element that are most interesting for us. In QFT, however, the generalization $q(t) \to \text{field}(t)$ with such time-ordered correlators will be absolutely crucial. Therefore, it will be very useful to us to "package" all such matrix elements with n factors into a *generating functional*

$$Z_{fi}[\underset{\substack{\uparrow \\ \text{source}}}{j(t)}] = \langle q_f, t_f|\, Te^{i\int_{t_i}^{t_f} dt\, j(t)Q(t)}\, |q_i, t_i\rangle, \tag{211}$$

from which

$$\langle q_f, t_f|\, T(Q(t_1)\dots Q(t_n))\, |q_i, t_i\rangle = \left.\frac{\delta^n Z_{fi}[j]}{i^n \delta j(t_1)\dots\delta j(t_n)}\right|_{j=0}. \tag{212}$$

The path-integral representation of the generating functional is

$$Z_{fi}[j(t)] = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q\; e^{iS+i\int_{t_i}^{t_f} dt\, j(t)\, q(t)}. \tag{213}$$

4.4 Functional differentiation

Above we have introduced *functional* derivatives, denoted by $\frac{\delta}{\delta j(t)}$, which are derivatives with respect to a *function*, not with respect to a c-number quantity. Instead of $\frac{\mathrm{d}x}{\mathrm{d}x} = 1$, it therefore holds that

$$\frac{\delta f(x)}{\delta f(x')} = \delta(x - x'), \tag{214}$$

with $\delta(x - x')$ being the "1" in the space of functions.
More formally, just like the limit of a finite difference defines a standard derivative, we can define

a functional derivative of the functional $F[\phi]$ by:

$$\int dx \frac{\delta F[\phi]}{\delta \phi(x)} \epsilon(x) + \mathcal{O}(\epsilon^2) = F[f+\epsilon] - F[f], \tag{215}$$

where $\epsilon(x)$ is understood as a small change of $\phi(x)$, i.e., it should be small everywhere (and possibly with compact support).

We denote functionals by angular brackets around their arguments (which are functions). Note that a functional maps a *function* to a *number*. The action $S[\phi]$ is a good example: It takes a function $\phi(x)$, and, by integrating its Lagrange density $\mathcal{L}(\phi)$ (which is itself a function of x through its dependence of ϕ and its derivative at each individual point), maps the whole configuration ϕ to a single number.

Based on the definition Eq. (215), we can derive that

$$\frac{\delta}{\delta \phi(y)} \phi(x) = \delta^d(x-y), \tag{216}$$

as follows (we'll write the derivation for $d = 1$): We consider the particular functional

$$F_\delta[f] = \int_{-\infty}^{\infty} dx\, \delta(x-y) f(x). \tag{217}$$

From the definition Eq. (215), it follows that

$$\int dx \frac{\delta F_\delta[f]}{\delta f(x)} \epsilon(x) = \int_{-\infty}^{\infty} dx\, \delta(x-y)(f(x)+\epsilon(x)) - \int_{-\infty}^{\infty} dx\, \delta(x-y) f(x) = \epsilon(y). \tag{218}$$

For the left-hand side to be equal to the right-hand-side, it must hold that

$$\frac{\delta F_\delta[f]}{\delta f(x)} = \delta(x-y). \tag{219}$$

Thus we have that

$$\delta(x-y) = \frac{\delta F_\delta[f]}{\delta f(x)} = \frac{\delta}{\delta f(x)} \int_{-\infty}^{\infty} dx\, \delta(x-y) f(x) = \frac{\delta}{\delta f(x)} f(y). \tag{220}$$

For functional derivatives, there is a product rule

$$\frac{\delta}{\delta \phi(x)} (F[\phi] G[\phi]) = \frac{\delta F[\phi]}{\delta \phi(x)} G[\phi] + F[\phi] \frac{\delta G[\phi]}{\delta \phi(x)}, \tag{221}$$

and a chain rule

$$\frac{\delta}{\delta \phi(x)} F[G[\phi]] = \int dy \left. \frac{\delta F[G]}{\delta G(y)} \right|_{G=G[\phi]} \frac{\delta G[\phi(y)]}{\delta \phi(x)}. \tag{222}$$

Mini-Exercise 11. Check that

$$-i \frac{\delta Z_{fi}[j(t)]}{\delta j(t_1)} \Big|_{j=0} = \int_{\substack{q(t_i)=q_i \\ q(t_f)=q_f}} \mathcal{D}q\, e^{i\,S[q]} q(t_1). \tag{223}$$

Solution.

$$-i\frac{\delta Z_{fi}[j(t)]}{\delta j(t_1)}\Big|_{j=0} = -i\frac{\delta}{\delta j(t_1)}\int_{\substack{q(t_i)=q_i\\q(t_f)=q_f}}\mathcal{D}q\, e^{iS+i\int dt\, j(t)\,q(t)}\Big|_{j=0} \tag{224}$$

$$= -i\int_{\substack{q(t_i)=q_i\\q(t_f)=q_f}}\mathcal{D}q\, i\left(\int dt\,\delta(t-t_1)q(t)\right)e^{iS+i\int dt\, j(t)\,q(t)}\Big|_{j=0} \tag{225}$$

$$= \int_{\substack{q(t_i)=q_i\\q(t_f)=q_f}}\mathcal{D}q\, q(t_1)e^{iS} \tag{226}$$

4.5 Projection onto the ground state at asymptotic times

Suggested reading for this section: Sec. 5.5 in Ryder; QFT I+II lecture notes from Heidelberg U. by Timo Weigand...

In QM, we typically care about transition states between states that are not the ground state. In QFT, for many problems, we are actually interested in a vacuum-to-vacuum transition amplitude, or the correlators evaluated in the ground state. The reason is twofold:

First, as we already had a glimpse of when we discussed the Casimir effect, already the vacuum (or ground state) is quite non-trivial in a QFT.

Second, we often care about a description of particle scattering events (e.g. at the LHC, or in a cosmic-ray shower, or in the IceCube neutrino detector...), where the particle(s) of interest are *created* (e.g., by collision), they interact and then they are destroyed (e.g. by detection).

The act of creation may be represented by a *source* and that of destruction (which is also, in some sense, a source). The boundary conditions of the problem may then be represented as in the following figure, where the vacuum at $t = -\infty$ evolves into the vacuum at $t \to +\infty$, via the creation, interaction and destruction of particles, through the agency of a source. We are thus

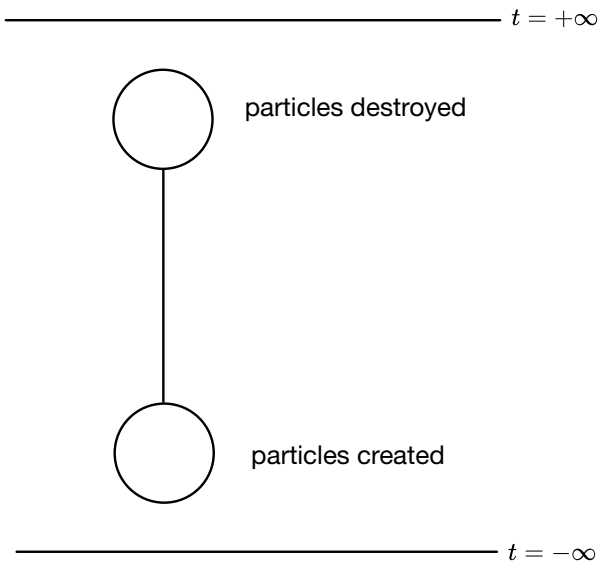


Figure 6: We sketch the idea behind the source acting over a finite amount of time.

interested in

$$\langle 0,\infty|0,-\infty\rangle\,. \tag{227}$$

How do we obtain it from

$$\big\langle Q',T'\big|Q,T\big\rangle = \int_{\substack{q(T')=Q'\\ q(T)=Q}} \mathcal{D}q\, e^{i\int_T^{T'} dt\,(L+Jq)}\ ? \tag{228}$$

We will consider this still in QM and then generalize to QFT.
We assume $J=J(t)$ and $J=0$ for $t<t''$ and $t>t'$, with $T<t''$ and $t'<T'$, i.e., the source is switched on after the asymptotic time T and switched off before the asymptotic time T'.
To distinguish whether the source is present or not, we write $\langle q_1,t_1|q_2,t_2\rangle^J$ if J is nonzero for (part of) the time inbetween t_1 and t_2, $t_2<t<t_1$ and otherwise, we leave out the J. Thus we consider

$$\big\langle Q',T'\big|Q,T\big\rangle = \int dq'\,dq\,\big\langle Q',T'\big|q',t'\big\rangle\big\langle q',t'\big|q,t\big\rangle^J\langle q,t|Q,T\rangle\,. \tag{229}$$

We can write

$$\big\langle Q',T'\big|q',t'\big\rangle = \big\langle Q'\big|\,e^{-iHT'}e^{iHt'}\,\big|q'\big\rangle\,, \tag{230}$$

where we used $|q,t\rangle = e^{-iHt}\,|q\rangle$.
We ultimately want to rewrite this into an expression containing the *ground state* and so it makes sense to use a complete set of *energy* eigenstates next. We write this as

$$\big|q',t'\big\rangle = \sum_n e^{iHt'}\,|n\rangle\big\langle n\big|q'\big\rangle = \underbrace{|0\rangle\big\langle 0\big|q'\big\rangle}_{\text{assuming } E_0=0} + \sum_{|n\rangle\neq|0\rangle} e^{iE_nt'}\,|n\rangle\big\langle n\big|q'\big\rangle\,. \tag{231}$$

For simplicity, we symbolize the energy eigenstates by the label "n" and write a sum over them. It doesn't matter for our derivation if they are actually continuous.
This leads to

$$\big\langle Q',T'\big|q',t'\big\rangle = \sum_n \langle n|\,e^{iE_n(t'-T')}\,|n\rangle\langle Q|n\rangle\langle q'|n\rangle \tag{232}$$

$$= \langle 0|0\rangle\big\langle Q'\big|0\big\rangle\big\langle 0\big|q'\big\rangle + \sum_{|n\rangle\neq|0\rangle}\langle n|\,e^{iE_n(t'-T')}\,|n\rangle\big\langle Q'\big|n\big\rangle\big\langle n\big|q'\big\rangle\,. \tag{233}$$

To isolate the contribution from the vacuum state, which is the one that we are interested in, we change $T'\to\infty(1-i\varepsilon)$ and $T''\to-\infty(1-i\varepsilon)$, with $\varepsilon>0$. Then,

$$e^{iE_n(t'-T')}\to e^{iE_n(t'-T'(1-i\varepsilon))} = e^{iE_n(t'-T')-\varepsilon T'E_n}\,. \tag{234}$$

The term $\sim\varepsilon$ is an *exponential suppression factor*. In the limit $T'\to\infty$, it results in a suppression of all states that are not the ground state. Thus, we obtain

$$\lim_{\substack{T'\to\infty(1-i\varepsilon)\\ T\to-\infty(1-i\varepsilon)}} \big\langle Q',T'\big|Q,T\big\rangle \sim \langle 0,\infty|0,-\infty\rangle\,, \tag{235}$$

or, in terms of the path integral:

$$\langle 0,\infty|0,-\infty\rangle = \mathcal{N}\int \mathcal{D}q\, e^{i\int_{-\infty}^{\infty}\left(L+Jq+\frac{1}{2}i\varepsilon q^2\right)}. \tag{236}$$

Note that we have dropped proportionality factors, that we just absorb in the overall normalization of the path integral. They drop out of physical observables.
Similarly, for vacuum expectation values of operators, we have a similar path integral. It doesn't have any boundary conditions; it is simply the path integral over *all* functions.

4.6 Path integral in quantum field theory

The path-integral formalism can be generalized from QM to QFT. It provides a framework that in many cases is easier to deal with (e.g., when formulating the theory of the strong interactions, QCD). Also, it provides a conceptually different way of thinking about QFT, where, just like in the multi-split-setup in QM, all *field configurations* are realized at the *same time* and interfere destructively and constructively.
We generalize from QM to QFT by

$$q(t) \to \phi(x) \tag{237}$$

$$p(t) \to \Pi(x) \tag{238}$$

$$j(t) \to j(x). \tag{239}$$

Herein, the arguments x of the fields and source are understood as spacetime arguments, i.e., ϕ etc. depend on spatial coordinates and time.
The main results from QM generalize and we obtain a generating functional

$$Z[j] = \int \mathcal{D}\Pi(x)\mathcal{D}\phi(x)\, e^{i\int d^4x \left(\Pi(x)\dot{\phi}(x) - (1-i\varepsilon)H + j(x)\phi(x)\right)}. \tag{240}$$

Because H is quadratic in $\Pi(x)$,

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi), \tag{241}$$

we can perform the Gaussian functional integral over $\Pi(x)$ and obtain

$$Z[j] = \int \mathcal{D}\phi(x)\, e^{i\int d^4x\, (\mathcal{L}(\phi) + j(x)\phi(x))} = \int \mathcal{D}\phi\, e^{iS + i\int j\phi}. \tag{242}$$

The generating functional generates *correlation functions*, which are expectation values of the field at different spacetime points. Just as in the case of Quantum Mechanics, these are time-ordered correlators.

$$\langle T\phi(x_1)\dots\phi(x_n)\rangle = i^n \frac{\delta}{\delta j(x_1)} \dots \frac{\delta}{\delta j(x_n)} Z[j] = \int \mathcal{D}\phi\, \phi(x_1)\dots\phi(x_n) e^{iS + i\int j\phi}. \tag{243}$$

(We do not care about normalization factors at this point.) These correlation functions tell us about the expectation value of the field in the vacuum and the *correlations* between field values at different points. When we remember that in the canonical formalism, $\phi(x)$ acting on the vacuum generates a particle, we can see that the correlators give us information on the amplitude for a particle to propagate from one point to another (for $\langle\phi(x)\phi(y)\rangle$) and the amplitudes for three, four etc particles to interact. Therefore, these correlation functions will later become important, when we describe particle scattering in quantum field theory. We will come back to them later and

develop a better and more detailed understanding of them.
Let us contrast classical field theory and quantum field theory: Classically, a single field configuration $\phi_{\text{class}}(x)$ is realized, for which

$$\left.\frac{\delta S}{\delta \phi}\right|_{\phi=\phi_{\text{class}}} = 0. \tag{244}$$

In QFT, all field configurations are realized at the same time. Each comes with a complex phase factor $e^{iS[\phi]}$, that results in destructive/constructive interference between field configurations. When S varies slowly, the interference is constructive, because "neighboring" configurations have nearly the same phase factor e^{iS}. Thus, we expect that a main contribution to the path integral actually comes from the classical field configurations. In contrast, field configurations far from the classical ones have a quickly varying phase factor e^{iS}, which oscillates between $+1$ and -1 quickly, and thus these cancel out or interfere destructively.
Overall, the path integral formalism provides a different, and highly useful, intuition for the physics of QFT, as well as a powerful formalism.

5 The quantum effective action $\Gamma[\Phi]$ and interacting quantum field theory

(See Gelis, Ryder, Padmanabhan ...)
In QFT, classical field configurations lose their meaning, because no *single* field configuration is ever realized (see the discussion above). However, the expectation value $\langle\phi\rangle = \int \mathcal{D}\phi\, \phi\, e^{iS[\phi]}$ has physical meaning, as do the n-point correlation functions $\langle\phi(x_1)\dots\phi(x_n)\rangle$.
Therefore, just like S produces the classical equations of motion, we would like to have a "quantum version" that gives the equations of motion for the expectation value $\langle\phi\rangle$.
We will call this (at this stage hypothetical) object $\Gamma[\Phi]$, where we have introduced the notation $\Phi = \langle\phi\rangle$, to clarify that the argument of Γ is *not* an individual classical field configuration, because Γ must actually arise from the path integral.

Let us define

$$e^{i\Gamma[\Phi]+i\int d^4x\, j(x)\Phi(x)} = \int \mathcal{D}\phi\, e^{iS[\phi]+i\int d^4x\, j(x)\phi(x)}, \tag{245}$$

because then $\Gamma[\Phi]$ appears analogously to $S[\phi]$ (in a complex exponential) and, if we would "switch off" quantum fluctuations, i.e., only have a contribution from the classical field configurations, then $\Gamma[\Phi] = S[\Phi]$.
In fact, this definition implies that

$$\Gamma[\Phi] = -i\ln Z[j] - \int d^4x\, j(x)\Phi(x), \tag{246}$$

i.e. $\Gamma[\Phi]$ is the *Legendre transform* of $\ln Z[j]$.

Mini-Exercise 12. Check that this expression for $\Gamma[\Phi]$ follows from our definition above.

Solution.

$$e^{i\Gamma[\Phi]+i\int d^4x\, j(x)\Phi(x)} = \underbrace{\int \mathcal{D}\phi\, e^{iS[\phi]+i\int d^4x\, \phi(x)j(x)}}_{Z[j]}$$

$$\implies \ln\left(e^{i\Gamma[\Phi]}e^{i\int d^4x\, j(x)\Phi(x)}\right) = \ln Z[j]$$

$$i\Gamma[\Phi] + i\int d^4x\, j(x)\Phi(x) = \ln Z[j]$$

$$\implies \Gamma[\Phi] = -i\ln Z[j] - \int d^4x\, j(x)\Phi(x).$$

From there, we obtain

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi} = -j(x). \tag{247}$$

This is the *quantum analogue* of the classical equations of motion. It gives us the equations of motion for the expectation value Φ.
Effective action as Legendre transform:
Let us be a bit more precise about the definition of $\Gamma[\Phi]$ as the Legendre transform of $\ln Z[j]$. First, we review what a Legendre transform of a function $f(x)$ is. (Useful reading for this is this

reference). Let the derivative of $f(x)$ be p, i.e.,

$$\frac{df}{dx} = p. \tag{248}$$

In our case, this would be a functional version of such an equation, namely $\frac{\delta}{\delta j(x)} Z[j] = \Phi$. We are now interested in obtaining a quantity that takes p as its argument, but contains the information on the function $f(x)$, i.e., we are looking for some $g(p)$ that arises from $f(x)$ and is in a precise correspondence to $f(x)$. We can derive from Eq. (248) that

$$d(xp - f(x)) = x\,dp + p\,dx - df = x\,dp. \tag{249}$$

Thus, by defining

$$g(p) = \sup_x \left(xp - f(x)\right), \tag{250}$$

we have that

$$\frac{dg}{dp} = x. \tag{251}$$

The supremum means that the right-hand side is evaluated at the value of x, where it takes its maximal value. This is how $g(p)$ depends only on p, and not also on x, as it would be, if we would leave out the supremum.
In our case, $g(p)$ is $\Gamma[\Phi]$, and we note that Eq. (246) should properly read

$$\Gamma[\Phi] = \sup_j \left(-i \ln Z[j] - \int d^4x j(x)\,\Phi(x)\right). \tag{252}$$

Let us finish this discussion with an example of the Legendre transform of a function. We consider $f(x) = x^2$. Then $px - f(x)$ is maximized at $x = p/2$, and thus $g(p) = (x \cdot p - f(x))\Big|_{x=p/2} = p^2/2 - p^2/4 = p^2/4$.
Comment: There's lots more to say about Γ and we will come back to it in QFT II to develop our formal and physical understanding of it further.

5.1 Effective potential in scalar $\lambda\phi^4$ theory

(See Gelis, Ryder, Peskin/Schröder, Padmanabhan)

We are now ready to take a first look at an interacting theory. We will learn about the effects of quantum fluctuations and will also encounter UV divergences again that we have to deal with.
We consider the simplest interacting theory, namely $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{12}\phi^4. \tag{253}$$

What is the physical meaning of the $\phi(x)^4$ term? Remember that $\phi(x)$, understood as an operator, when acting on the vacuum, produces a one-particle state with the particle localized at x. Thus, if we act with $\phi(x)$ four times, we describe four particles, all localized at x. This is clearly what we need in order to describe *interacting* particles, because, to describe interactions, we must have several particles, not just a single one.

Note also that the interaction is *local*, i.e., particles interact at a single point in spacetime. (An example for a nonlocal interaction term would be $\int d^4x \int d^4y\, \phi(x)^2\phi(y)^2$. There is no QFT with such interactions, because causality would be impossible to maintain in such a theory.) Local interactions are what we observe in nature (e.g., at particle colliders, interactions happen when particles meet at a point, they do not happen over a distance) and thus we use local interaction terms in Lagrangians.
We exclude the ϕ^3-term by a $\phi \to -\phi$ symmetry ($\mathbb{Z}_2$ symmetry) and we neglect other interactions (ϕ^6, $\partial_\mu\phi\partial^\mu\phi\,\phi^2$ etc.) for now. ϕ^4 is an interaction term, because it yields a non-linear term in the equations of motion, i.e., instead of a freely propagating wave-like solution, we have *self-interactions*. In terms of the corresponding particles, we can have *scattering* of the particles off each other.

The path integral for an interacting theory is complicated. It is no longer, as it would be for the free theory, a Gaussian integral that we can do exactly. Therefore, we will consider an expansion for it, in which the leading non-trivial correction is also obtained from a term quadratic in the fields.
We expand around $\phi_{\text{cl}} = \langle\phi\rangle$. Because we expect that the ground state of the theory respects translation invariance, $\phi_{\text{cl}} = const.$ This constant may be zero, but, and this will be the more interesting case, it may also be non-zero. In this case, the $\mathbb{Z}_2$-symmetry is broken *spontaneously*, i.e., there is a symmetry of the Lagrangian that the ground state breaks. This also happens in the Higgs sector of the Standard Model and is the mechanism through which the elementary particles in the SM acquire their mass[9].

In $Z[j]$, we can perform a *shift* of the integration variable,

$$Z[j] = \int \mathcal{D}\phi\, e^{i(S[\phi]+\int j\phi)} \underset{\substack{\uparrow\\ \phi=\phi_{\text{cl}}+\varphi}}{=} \int \mathcal{D}\varphi\, e^{iS[\phi_{\text{cl}}+\varphi]+i\int j(\phi_{\text{cl}}+\varphi)}. \tag{254}$$

Now, we can expand S in φ. Intuitively, we can imagine that we are aiming at calculating the impact of small quantum fluctuations around ϕ_{cl}, i.e., we account for the effect of field configurations close to ϕ_{cl}.
We set $j = \left.\frac{\delta S}{\delta\phi}\right|_{\phi=\phi_{\text{cl}}}$, so that the source only sources the classical field. Then,

$$S[\phi_{\text{cl}}+\varphi] + \int d^4x\, j(\phi_{\text{cl}}+\varphi) \tag{255}$$

$$= S[\phi_{\text{cl}}] + \int d^4y\, \left.\frac{\delta S}{\delta\phi(y)}\right|_{\phi=\phi_{\text{cl}}} \varphi(y) + \frac{1}{2}\int d^4y\, d^4z\, \varphi(y)\, \left.\frac{\delta^2 S[\phi]}{\delta\phi(y)\delta\phi(z)}\right|_{\phi=\phi_{\text{cl}}} \varphi(z) + \ldots + \int d^4x\, j(\phi_{\text{cl}}+\varphi). \tag{256}$$

Because of $j = \left.\frac{\delta S}{\delta\phi}\right|_{\phi=\phi_{\text{cl}}}$, the term linear in φ and the source term with φ cancel. We evaluate the $\delta^2 S$ term as a mini exercise.

[9] The QCD bound states, such as the proton and neutron, acquire most of their mass through another spontaneous symmetry breaking due to the strong-coupling regime that QCD enters at low energies.

Mini-Exercise 13. Evaluate

$$\frac{1}{2}\int d^4y\, d^4z\, \varphi(y)\, \frac{\delta^2 S[\phi]}{\delta\phi(y)\delta\phi(z)}\bigg|_{\phi=\phi_{\rm cl}} \varphi(z). \tag{257}$$

Solution. We obtain ($\Box = \partial_\mu\partial^\mu$)

$$\begin{aligned}
&\frac{\delta}{\delta\phi(y)}\frac{\delta}{\delta\phi(z)}\int d^4x\, \left(-\frac{1}{2}\phi(x)\Box\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{12}\phi^4(x)\right)\bigg|_{\phi=\phi_{\rm cl}} \\
&\qquad = \int d^4x\, \Big(-\frac{1}{2}\delta^4(x-y)\Box\delta^4(x-z) - \frac{1}{2}\delta^4(x-z)\Box\delta^4(x-y) \\
&\qquad\quad - m^2\delta^4(x-y)\delta^4(x-z) - \lambda\phi^2(x)\delta^4(x-y)\delta^4(x-z)\Big)\bigg|_{\phi=\phi_{\rm cl}} \\
&\qquad = -\Box\delta^4(y-z) - m^2\delta^4(y-z) - \lambda\phi_{\rm cl}^2\delta^4(y-z).
\end{aligned}$$

We can write this as $-(\Box + V''(\phi_{\rm cl}))\delta^4(y-z)$, where $V'' = \frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}$. Thus,

$$Z[j] = \int \mathcal{D}\varphi\, e^{iS[\phi_{\rm cl}] - i\frac{1}{2}\int d^4y\, d^4z\, \varphi(y)\left[\left(\Box + V''(\phi_{\rm cl})\right)\delta^4(y-z)\right]\varphi(z) + i\int d^4x\, j\phi_{\rm cl}} \tag{258}$$

$$= \underbrace{e^{iS[\phi_{\rm cl}] + i\int d^4x\, j\phi_{\rm cl}}}_{\substack{\text{independent of } \varphi, \\ \text{can therefore be pulled out} \\ \text{of the path intergral.}}} \int \mathcal{D}\varphi\, e^{-i\frac{1}{2}\int d^4y\, \varphi(y)\left(\Box + V''(\phi_{\rm cl})\right)\varphi(y)}. \tag{259}$$

Now we need that (for a proof, see the exercises)

$$\int \mathcal{D}\varphi\, e^{-\frac{1}{2}\phi(x)A\phi(x)} = (\det A)^{-\frac{1}{2}}, \tag{260}$$

$$\det A = e^{\operatorname{tr}\ln A}, \tag{261}$$

$$(\det A)^{-\frac{1}{2}} = e^{-\frac{1}{2}\operatorname{tr}\ln A}. \tag{262}$$

This helps us, because this expression also continues to hold if the "matrix" A is infinitely large, i.e., we use that

$$\int d^4x \int d^4y \varphi(x)\left(\Box + V''\right)\delta^4(x-y)\varphi(y) = \int d^4x \varphi(x) A(x,y)\varphi(y) \leftarrow \sum_i\sum_j \varphi(x_i)A_{ij}\varphi(x_j). \tag{263}$$

Thus,

$$Z[j] = e^{iS[\phi_{\rm cl}] + i\int j\phi_{\rm cl}} e^{-\frac{1}{2}\operatorname{tr}\ln\left(-i\left(\Box + V''(\phi_{\rm cl})\right)\right)} + \dots \tag{264}$$

and finally, we obtain the expression for the *one-loop effective action* for a constant $\phi_{\rm cl}$:

$$\Gamma[\phi_{\rm cl}] = S[\phi_{\rm cl}] + \frac{i}{2}\operatorname{tr}\ln\left(\Box + V''(\phi_{\rm cl})\right), \tag{265}$$

where we first calculate $-i\ln Z[j]$ to obtain Γ, and where we can drop the "i" inside the ln, because it only produces a constant addition to Γ which is field-independent and therefore irrelevant for the physics.

This expression makes sense in that $\Gamma[\phi_{\rm cl}] = S[\phi_{\rm cl}]$ to leading order. If we reinstate $\hbar$, then the tr ln-term is $\sim \hbar$, so it encodes the leading-order quantum correction. This term is called the 1-loop term, because the tr implies an integration over momenta (the eigenvalues of $\Box$) of quantum fluctuations. When we introduce the expansion in terms of Feynman diagrams later, we will see that we will denote such an integration by a closed loop.
In the case of $\phi_{\rm cl} = const$ that we consider here, $\Gamma[\phi_{\rm cl}] = \int d^4x\, V_{\rm eff}[\phi_{\rm cl}]$, because the kinetic term vanishes for this case. $V_{\rm eff}$ is called the *effective potential*. Our remaining task is to evaluate

$$\frac{i}{2}\,\mathrm{tr}\ln\left(\Box + V''(\phi_{\rm cl})\right) = \frac{i}{2}\int d^4x \int \frac{d^4p}{(2\pi)^4}\,\ln\left(-p^2 + \underbrace{m^2 + \lambda\phi_{\rm cl}^2}_{\text{constants}}\right). \tag{266}$$

$\int d^4x$ just becomes a factor of spacetime volume that also appears before $V_{\rm eff}$. These drop put of $V_{\rm eff}$. To evaluate the integral, we do several things:

- we note that the integral is divergent at large p^2. This is again a consequence of us having simply assumed that our QFT is valid up to arbitrarily small distances (i.e., arbitrarily high momenta).
- we point out that m^2, which we have been calling "mass", does not actually correspond to a measurable quantity. Rather, $\frac{\partial^2}{\partial\phi_{\rm cl}^2}V\Big|_{\phi_{\rm cl}=0} = m^2_{\rm Phys}$ is the actual mass that is associated to the field. If we could turn off $\hbar$, then $m^2_{\rm Phys}$ would correspond to the mass, but in nature, we *cannot* turn off $\hbar$ (or, more physically, we cannot turn off quantum fluctuations).

These two observations combined lead us to the following ideas:

1) The momentum integral should be regularized. The most intuitive way is through a cutoff Λ in the momentum integral, although there are other ways of regularizing, e.g., dimensional regularization, that we encounter later.

2) The cutoff (or more generally the regularization) just parametrizes our ignorance of the correct physics at high momenta (small distances).
The *physical results* of measurements (e.g., of the mass of a particle or the strength with which particles interacting) do not care about whether or not we do or do not understand the UV physics. Therefore,

$$\frac{\partial^2}{\partial\phi_{\rm cl}^2}V_{\rm eff}(\phi_{\rm cl})\bigg|_{\phi_{\rm cl}=0} = m^2_{\rm Phys} \tag{$\star$1}$$

$$\frac{\partial^4}{\partial\phi_{\rm cl}^4}V_{\rm eff}(\phi_{\rm cl})\bigg|_{\phi_{\rm cl}=0} = 2\lambda_{\rm Phys} \tag{$\star$2}$$

must not depend on the cutoff Λ (or whatever other parameter determines our regularization). To achieve that ($\star$1) and ($\star$2) are independent from Λ, we are led to the conclusion that m^2 and λ must be functions of Λ. We are free to make them that, because they are not physical parameters that we could measure.
Therefore, we will implement the procedure of regularization and renormalization. Note that "renormalization" is a misnomer, because it suggests that we have already normalized the theory and now we need to "re"-normalize all parameters by huge (potentially even infinite) shifts. This is a confusing view of the actual procedure, where we normalize the measurable

parameters once.

We will later introduce the procedure of renormalization more formally and comprehensively. For now, our goal is to understand the key idea behind it.

For literature focused on the conceptual idea without a lot of formalism, check out chapter III in Zee's *Quantum Field Theory in a nutshell.*

In practice, we now need to regularize. There are different methods of regularization, e.g., zeta-function regularization (which is probably the most abstract of the regularization methods typically used), dimensional regularization (which we will encounter and use later in the course, and which is a very common method in high-energy physics) and cutoff-regularization. We already encountered cutoff-regularization when we considered the Casimir effect, and we will use it again here. Arguably, in cutoff-regularization, it is easiest to understand what is going on.
To perform the regularization, we perform a *Wick-rotation* to Euclidean signature, i.e., we send

$$t \to i\tau, \tag{267}$$

where τ is Euclidean "time", i.e., the "time" coordinate in a space with a metric with all negative signs. Then, $p^2 \to -p_E^2$, and

$$\int \frac{d^4p}{(2\pi)^4} \ln\left(-p^2 + m^2 + \lambda\phi_{\rm cl}^2\right) \to \int \frac{d^4p_E}{(2\pi)^4} \ln\left(p_E^2 + m^2 + \lambda\phi_{\rm cl}^2\right). \tag{268}$$

The reason for doing a Wick-rotation is that we can now successfully introduce a cutoff, i.e., limit $p_E^2 < \Lambda^2$. In Minkowski signature, $p^2 < \Lambda^2$ does not effectively work as a cutoff, because $p^2 = p_0^2 - \vec{p}^2$, and thus $p^2 < \Lambda^2$ still allows arbitrarily high energies p_0^2, as long as they come with an arbitrarily high spatial momentum $\vec{p}_i^2$.
Thus,

$$\begin{aligned}
\int \frac{d^4p_E}{(2\pi)^4} \ln\left(p_E^2 + m^2 + \lambda\phi_{\rm cl}^2\right) &= \int_0^\infty \frac{dp_E^2}{32\pi^2} p_E^2 \ln\left(p_E^2 + m^2 + \lambda\phi_{\rm cl}^2\right) \\
&\to \int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln\left(p_E^2 + m^2 + \lambda\phi_{\rm cl}^2\right) \\
&\to \int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln\left(\frac{p_E^2 + m^2 + \lambda\phi_{\rm cl}^2}{\Lambda^2}\right).
\end{aligned} \tag{269}$$

In the first step, we have used that the integral is rotationally symmetric in momentum space. In the second step, we have introduced a regularization, that we ultimately would like to remove again, i.e., we aim at sending $\Lambda^2 \to \infty$ to recover the original integral. In the meantime, we are keeping in mind that the high-momentum part of the integral comes from a regime where we cannot be sure whether our theory actually describes nature – we have simply extrapolated our theory to arbitrarily high momenta (arbitrarily small distances). Thus, Λ^2 can be given a physical interpretation in the sense that it cuts off the part of the momentum integral, where some "new physics" may be, that our theory does not account for and therefore does not adequately describe this regime.
In the very last step, we have noted that the argument of the ln is dimensionful (which it should not be), and have therefore divided by Λ^2. This we have achieved by subtracting $\int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln\left(\Lambda^2\right)$ from the previous expression. We are free to do this, because there is no dependence on the *field* in this. We are just subtracting a field-independent constant, which does not contribute to the

equations of motion (and just shifts the value of the ground-state energy, but has no effect on measurements, because we can only measure differences in energies.
Now we have an expression that we can evaluate and interpret. We will learn several important consequences of quantum fluctuations, that are not specific to this particular theory, but rather generic features of many QFTs.
We obtain

$$\int_0^{\Lambda^2} \frac{dp_E^2}{32\pi^2} p_E^2 \ln\left(\frac{p_E^2+m^2+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right) = \frac{1}{4\cdot 32\pi^2}\Big[\Lambda^2\left(2m^2-\Lambda^2+2\lambda\phi_{\rm cl}^2\right)+2\Lambda^4\ln\left(\frac{m^2+\Lambda^2+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right) -2\left(m^2+\lambda\phi_{\rm cl}^2\right)^2\left(\ln\left(\frac{m^2+\lambda\phi_{\rm cl}^2+\Lambda^2}{\Lambda^2}\right)-\ln\left(\frac{m^2+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right)\right)\Big]. \tag{270}$$

From there, by collecting all prefactors, and adding the classical contribution, we obtain the effective potential

$$V_{\rm eff}(\phi_{\rm cl}) = \frac{1}{4\cdot 32\pi^2}\Big[\Lambda^2\left(2m^2-\Lambda^2+2\lambda\phi_{\rm cl}^2\right)+2\Lambda^4\ln\left(\frac{m^2+\Lambda^2+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right) -2\left(m^2+\lambda\phi_{\rm cl}^2\right)^2\left(\ln\left(\frac{m^2+\lambda\phi_{\rm cl}^2+\Lambda^2}{\Lambda^2}\right)-\ln\left(\frac{m^2+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right)\right)\Big]+\frac{m^2}{2}\phi_{\rm cl}^2+\frac{\lambda}{12}\phi_{\rm cl}^4. \tag{271}$$

This is a complicated and somewhat lengthy expression, so there is lots to unpack. In fact, there are also several important physical aspects to learn from this expression that we will go through in detail below:

1. We see that the effective potential contains terms that have the same field dependence as the classical terms, but come with prefactors that will diverge if we take $\Lambda^2 \to \infty$. One example is the first term in the first line of Eq. (271), which is $2\Lambda^2\,\lambda\,\phi_{\rm cl}^2$. These are the terms that we will have to deal with through renormalization and that have caused some confusion about QFT, in particular in the earlier years of the development of QFT. Here, we will not discuss renormalization in detail, nor introduce all technical details of the renormalization procedure; we will just take a look at the main ideas. Later in the course, we will study renormalization in more detail.

2. We also see that, upon expanding the ln-terms, we will obtain higher powers of $\phi_{\rm cl}$, e.g., $\phi_{\rm cl}^6$ or $\phi_{\rm cl}^8$. This is another *generic feature of QFTs: quantum fluctuations generate new interactions*. This result is not specific to ϕ^4 theory, but appears in virtually all other QFTs. A well-known, and physically really interesting, example, is Quantum Electrodynamics: one of the phenomenologically crucial features of classical electrodynamics is that electromagnetic waves do not interact with each other, i.e., the equations of motion are linear, and accordingly photons do not interact. If this was different, and the equations would have strong non-linearities, we would not be able to observe distant galaxies (nor would one be able to see a black-board from across a lecture room, if the non-linearity was really strong). It turns out that when we integrate quantum fluctuations of the electron, a four-photon-interaction term is generated. Therefore, the quantum version of electrodynamics is fundamentally different from the classical version – at least in principle. In practice, we are saved by the fact that the photon-photon-interaction term comes with a tiny prefactor, such that one needs extremely high electromagnetic fields to observe the non-linearities. High-intensity lasers are

constructed with the goal (among other goals, of course), to observe these terms for the first time.

The corresponding action, encoding these terms, can be obtained from Quantum Electrodynamics along exactly the same line as our calculation for the scalar field and is called the *Euler-Heisenberg* effective action.

3. It is not obvious from the above expression, but, after we have taken care of the divergences, the resulting potential is generically no longer minimized at the point $\phi_{\rm cl} = 0$. Instead, a *non-zero expectation value* of ϕ permeates the vacuum. This is called *spontaneous symmetry breaking* (i.e., the ground state does not realize a symmetry of the action; in this case, the $\mathbb{Z}_2$ symmetry of the action, under which $\phi \to -\phi$, is not realized by the ground-state, because the only field configuration that realizes that symmetry is $\phi = 0$, but we will have that $\phi_{\rm cl} \neq 0$ at the minimum of the effective potential.

   Spontaneous symmetry breaking is a crucial ingredient of the Standard Model of particle physics, where the Higgs sector has a spontaneously broken symmetry. The resulting non-zero vacuum expectation value of the Higgs field that permeates the vacuum results in mass-generation for the fermions as well as some of the gauge bosons of the Standard Model.

   Here, we will see that quantum fluctuations can have the effect to spontaneously break a symmetry that is realized classically.

To see all these results, we will now analyze the $\phi_{\rm cl}^2$, $\phi_{\rm cl}^4$ and $\phi_{\rm cl}^6$ terms separately and then turn to the analysis of the full expression for the finite part of the effective potential.
We first note that any terms that are of zeroth order in the field, whether they are divergent or not, can simply be ignored. They contribute to a (possible infinite) shift of what we declare to be "zero energy", but (under the assumption of gravity not existing, that we operate under in this course), this shift can be ignored, because we can only measure energy differences.
To isolate individual powers of $\phi_{\rm cl}$, we expand the logarithmic terms as follows, which we can do, because we assume that $\Lambda^2 \gg m^2, \phi_{\rm cl}^2$.

$$\ln\left(\frac{\Lambda^2 + m^2 + \lambda\phi_{\rm cl}^2}{\Lambda^2}\right) = \ln\left(1 + \frac{m^2 + \lambda\phi_{\rm cl}^2}{\Lambda^2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{m^2 + \lambda_{\rm cl}^2}{\Lambda^2}\right)^n, \tag{272}$$

as well as

$$\ln\left(\frac{m^2 + \lambda\phi_{\rm cl}^2}{\Lambda^2}\right) = \ln\left(\frac{m^2}{\Lambda}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{\lambda\phi_{\rm cl}^2}{m^2}\right)^n. \tag{273}$$

Thus we obtain

$$\begin{aligned} V_{\rm eff}\Big|_{\phi_{\rm cl}^2} = & -\frac{1}{8\cdot 32\pi^2}\left(\Lambda^2 2\lambda + 2\Lambda^4\frac{\lambda}{\Lambda^2} - 2\cdot 2m^2\lambda\frac{m^2}{\Lambda^2} - 2m^4\frac{\lambda}{\Lambda^2} - 2\cdot 2m^2\lambda\,\ln\frac{m^2}{\Lambda^2} - 2m^4\frac{\lambda}{m^2}\right) \\ & + \frac{m^2}{2}. \end{aligned} \tag{274}$$

The terms in the first line originate from quantum fluctuations, the term in the second line is the classical contribution. We observe that *all terms in the first line are* $\sim \lambda$, i.e., interactions are necessary in order for quantum fluctuations to have an effect on the potential. Heuristically, we can imagine quantum fluctuations as virtual particles. A cloud of "virtual" particles is always there, even in the ground state, and it interacts with any real particles, and can affect their properties

(e.g., their mass). However, to have such an effect, virtual and real particles need to interact with each other, and thus, there must be an interaction term in the theory for the effect to be present. We see that there are some contributions to the $\phi_{\rm cl}^2$-term which are finite, even in the limit $\Lambda^2 \to \infty$, so the effect of quantum fluctuations that changes the potential, is there *irrespective of whether or not UV divergences are present.*

However, there are also contributions that would diverge in the limit $\Lambda^2 \to \infty$. To deal with them, we remember that *we cannot measure the classical term and the effect of quantum fluctuations separately, we can only measure their combination.* Thus, m^2 is so far an unspecified parameter. If we thus require that

$$V_{\rm eff}\Big|_{\phi_{\rm cl}^2} = \frac{m_{\rm phys}^2}{2}\phi_{\rm cl}^2, \tag{275}$$

with a finite $m_{\rm phys}^2$ (that corresponds to the value an experiment would measure), we see that we have to make the parameter m^2 dependent on Λ in such a way that their combination gives $m_{\rm phys}^2$. This, in essence, is the idea of *renormalization*, that we develop in more detail later in the course and in QFT II. Note that here we're not covering all steps of the renormalization; we're just sketching out the main ideas.

Next, we turn to the $\phi_{\rm cl}^4$ term. To isolate it from the ln-terms, it is useful to use of a slightly different expansion, namely

$$\ln\left(\frac{m^2+\Lambda^2+\lambda\phi_{\rm cl}^2}{\Lambda^2}\right) = \ln\left(1+\frac{m^2}{\Lambda^2}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{\lambda\phi_{\rm cl}^2}{m^2+\Lambda^2}\right)^n. \tag{276}$$

Using this, we obtain

$$\begin{aligned} V_{\rm eff}\Big|_{\phi_{\rm cl}^4} &= -\frac{1}{8\cdot 32\pi^2}\left[2\Lambda^4\frac{\lambda^2}{m^2+\Lambda^2} - 2\lambda^2\left(\ln\left(1+\frac{m^2}{\Lambda^2}\right) - \ln\frac{m^2}{\Lambda^2}\right)\right. \\ &\quad \left. +2m^2\cdot 2\lambda\left(\frac{\lambda}{m^2+\Lambda^2} - \frac{\lambda}{m^2}\right) - 2m^4\left(-\frac{\lambda^2}{2\left(m^2+\Lambda^2\right)^2} + \frac{\lambda^2}{2m^4}\right)\right] + \frac{\lambda}{12}. \end{aligned} \tag{277}$$

We again see that the terms from quantum fluctuations are proportional to the interaction, in this case, they are $\sim \lambda^2$. There are terms that remain finite in the limit $\Lambda^2 \to \infty$, so we see that quantum fluctuations also change the quartic term (i.e., the *strength* of the interaction), in the effective potential.

Then, there are again divergent terms. We absorb them by defining the physical interaction strength

$$V_{\rm eff}\Big|_{\phi_{\rm cl}^4} = \frac{\lambda_{\rm phys}}{12}. \tag{278}$$

This is possible, because we can adjust λ to depend on Λ in such a way that the divergences in $\lambda_{\rm phys}$ cancel. Again, we are not spelling out the details here and are not showing that we can really do renormalization consistently by using just the two parameters m and λ. Instead, we are just sketching out the main ideas and concepts.

In general, we will call theories in which we can absorb divergences within the existing mass/coupling parameters *renormalizable*. This means that in such theories, the unknown physics at high momenta (beyond the cutoff), affects these couplings (in our case, mass and quartic coupling). Therefore, we cannot know the physical values $m_{\rm phys}^2$ and $\lambda_{\rm phys}$, because the UV physics that they depend on is not known. They parameterize our ignorance about the UV properties of the theory.

However, the rest of the coupling parameters (in our case, this will be λ_6 of ϕ^6 etc, as we will see below), are *calculable* and *independent of the UV physics*.
Renormalizability thus means that we have a theory with finitely many free parameters – a *predictive* theory.
Usually, in QFT courses, renormalization is discussed later, and we will also come back to discuss it in more detail. However, taking a look at it early on is useful not just to understand what the effect of quantum fluctuations – the existence of which is the key difference between a classical field theory and a quantum field theory – actually is, but also in order to check whether they result in a theory that is not predictive at the quantum level, because it has a infinite number of free parameters.[10]
Now let us consider the $\phi^6_{\rm cl}$ terms. If these are also divergent, then we are in trouble, because we have used up the free parameters of our classical action, m^2 and λ, to absorb the divergences.
We obtain

$$\begin{aligned} V_{\rm eff}\Big|_{\phi^6_{\rm cl}} &= \frac{-1}{8\cdot 32\pi^2}\Bigg[2\Lambda^4\frac{\lambda^3}{3\left(m^2+\Lambda^2\right)^3} - 2m^4\left(\frac{\lambda^3}{3(m^2+\Lambda^2)^3} - \frac{\lambda^3}{3(m^2)^3}\right) \\ &\quad -2m^2\cdot 2\lambda\left(-\frac{\lambda^2}{2(m^2+\Lambda^2)^2} + \frac{\lambda^2}{2(m^2)^2}\right) - 2\lambda^2\left(\frac{\lambda}{m^2+\Lambda^2} - \frac{\lambda}{m^2}\right)\Bigg] + 0. \end{aligned} \tag{279}$$

The last term is the classical contribution, which is zero. The quantum contribution is non-zero, i.e., quantum fluctuations generate new interactions.
We observe that the limit $\Lambda^2 \to \infty$ is *finite* (this is if we keep m^2 and λ fixed; a proper discussion of course needs to consider the ϕ^6 term in terms of renormalized couplings); there are no divergences at this order in the field. We obtain

$$V_{\rm eff}\Big|_{\phi^6_{\rm cl}} \underset{\Lambda^2\to\infty}{\longrightarrow} -\frac{-1}{12\cdot 32\pi^2}\frac{\lambda^3}{m^2}. \tag{280}$$

Therefore, quantum fluctuations generate a *calculable* ϕ^6 interaction (and similarly, higher-order interactions). These are independent of the presence of divergences in the quadratic and the quartic term.
This result, that quantum fluctuations generate new interactions, with finite, calculable coefficients, is general and is not tied to whether or not there are divergences in the theory.

Finally, let us isolate the finite piece in the effective potential without Taylor-expanding in the field. We assume that we have previously taken care of the divergences and they result in a quadratic and quartic term with undetermined coefficients and we can write the finite part in terms of $m_{\rm phys}$ and $\lambda_{\rm phys}$.
We have

$$V_{\rm eff}\Big|_{\rm finite} = \frac{m^2_{\rm phys}}{2}\phi^2_{\rm cl} + \frac{\lambda_{\rm phys}}{12}\phi^4_{\rm cl} + \frac{1}{2\cdot 32\pi^2}\left(m^2_{\rm phys} + \lambda_{\rm phys}\phi^2_{\rm cl}\right)^2 \ln\left(\frac{m^2_{\rm phys} + \lambda_{\rm phys}\phi^2_{\rm cl}}{\mu^2}\right). \tag{281}$$

[10] There is much more to say and understand here, part of which we will cover in QFT II. It has to do with understanding non-renormalizable theories as effective field theories, and with understanding that renormalizable theories do not automatically make sense at all scales. Keywords that we will cover later (in QFT II), are asymptotic safety, asymptotic freedom, effective field theories, the Renormalization Group and Landau poles. This will be relevant for the quantum physics of QCD, QED and the Standard Model as a whole, as well as for numerous condensed-matter models. It will even touch on ideas for a quantum theory of gravity.

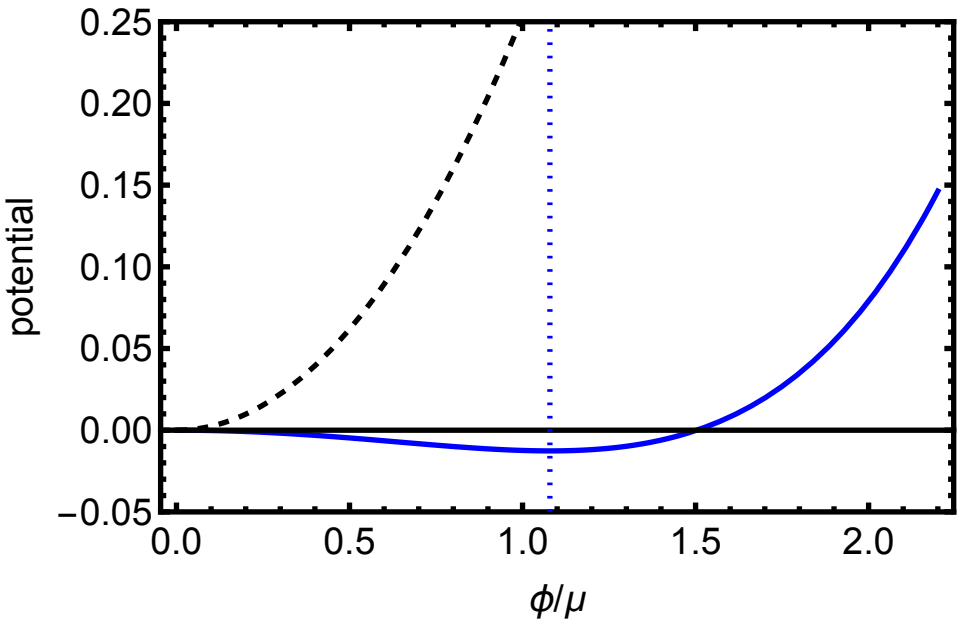


Figure 7: We show the finite part of the effective potential (for $\lambda_{\rm phys} = 0.1$ and $m^2/\mu^2_{\rm phys} = 0.7^2$), in blue. The blue dotted vertical line indicates where the minimum of the potential lies. The black dashed line is the classical potential with the same values of mass and coupling.

For small enough $\phi^2_{\rm cl}$, the ln is negative. Therefore the potential takes the form shown in Fig. 7. Whereas the classical theory has $\phi = 0$ as a solution, quantum fluctuations introduce a *non-zero* value of $\phi_{\rm cl}$ at the minimum of the potential (for some values of parameters of the theory). In those cases, the $\mathbb{Z}_2$-symmetry of the classical action is *broken spontaneously* in the ground state. This means that the symmetry of the action is not realized by the ground state, because $\phi_{\rm cl} = 0$, which is the only field configuration that realizes this symmetry, is not a minimum.
Instead, we can evaluate where the two possible minima lie. We take the derivative of $V_{\rm eff}\Big|_{\rm finite}$ with respect to $\phi_{\rm cl}$ and set it to zero. This equation determines extrema. $\phi_{\rm cl} = 0$ is a solution, but does *not* correspond to a minimum for all values of parameters. Instead, the remaining two solutions are

$$\phi_{\rm cl} = \pm\sqrt{\frac{\mu^2 e^{-\mu^2/2} - m^2_{\rm phys}}{\lambda_{\rm phys}}} \neq 0. \tag{282}$$

The choice between the two minima that the vacuum has to "spontaneously make", is the act of spontaneous symmetry breaking. It leads to a non-zero expectation value of the field that permeates the vacuum.

6 Gauge fields

In this chapter, we will learn how the existence of gauge fields (like the electromagnetic field) necessarily follows from considering local symmetries, i.e., symmetries where the symmetry transformation depends on the spacetime point. This is one of the profound connections between different fundamental concepts that we encounter in QFT.
It is useful to know the formulation of electrodynamics in terms of a gauge field that is a 4-vector, $A^\mu(x)$, the corresponding field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the gauge transformation $A_\mu \to A_\mu{}' = A_\mu - \partial_\mu \Lambda$ which leaves physics invariant. We will also use polarization vectors. If you need to refresh your memory of these concepts, check out

- Chapter 4 of David Tong's lectures on electrodynamics (online)
- Chapter 5 of Arthur Hebecker's lectures on electrodynamics (online)

To start with, we go back to Noether's theorem and evaluate the conserved current that is associated to the $U(1)$ symmetry of a complex scalar field, as you did in one of the exercises.

We consider a function $\phi(x)$ that takes values in the complex numbers. The Lagrangian has to be real, because the action has to be real and thus

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \tag{283}$$
$$= |\partial_\mu \phi|^2 - m^2 |\phi|^2. \tag{284}$$

The Lagrangian is invariant under a global $U(1)$ symmetry group of phase rotations

$$\phi \to e^{i\alpha} \phi, \tag{285}$$

i.e., α is a number, not a function of spacetime coordinates. The infinitesimal version is

$$\phi \to \phi' = e^{i\varepsilon}\phi = \phi + i\varepsilon\phi + \mathcal{O}(\varepsilon^2) \tag{286}$$
$$\phi^\dagger \to \phi^{\dagger\prime} = e^{-i\varepsilon}\phi^\dagger = \phi^\dagger - i\varepsilon\phi^\dagger + \mathcal{O}(\varepsilon^2). \tag{287}$$

We can work either infinitesimally or with the non-infinitesimal phase rotation to show that the Lagrangian is invariant:

$$\mathcal{L}' = \partial_\mu \cancel{e^{i\alpha}} \phi \partial^\mu \cancel{e^{-i\alpha}} \phi^\dagger - m^2 \phi^\dagger \cancel{e^{-i\alpha} e^{i\alpha}} \phi = \mathcal{L}. \tag{288}$$

Here, we could cancel the factors $e^{\pm i\alpha}$, because they are constants and can be pulled past the partial derivatives.
The symmetry relates the two real scalar fields in ϕ, namely its real and imaginary part, to each other. We can see this by writing

$$\phi(x) = \phi_1(x) + i\,\phi_2(x), \tag{289}$$

where $\phi_{1,2}(x)$ are both real. Then we have a Lagrangian that is given by

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi = \partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2 \phi_1^2 - m^2 \phi_2^2. \tag{290}$$

The symmetry between the two real scalar fields holds, because their masses are both the same. If we start from a Lagrangian for two independent scalar fields and we would like a U(1) symmetry, we simply have to choose their masses equal. (Similarly, had we introduced interactions for ϕ (and correspondingly $\phi_{1,2}$), the interaction terms would have to be built out of $\phi\,\phi^\dagger$, i.e., $\phi_1^2+\phi_2^2$, and thus, e.g., the quartic couplings for ϕ_1 and ϕ_2 would have to be equal to obtain the symmetry, and there would also have to be a $\phi_1^2\,\phi_2^2$-interaction.
The conserved current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\varepsilon\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\delta_\varepsilon\phi^\dagger \tag{291}$$

$$= (\partial^\mu\phi^\dagger)i\phi + (\partial^\mu\phi)(-i\phi^\dagger) \tag{292}$$

$$= -i(\phi^\dagger\overleftrightarrow{\partial^\mu}\phi). \tag{293}$$

(we saw above that there is no boundary term that arises in $\mathcal{L}$.) Here we have defined

$$A\overleftrightarrow{\partial^\mu}B = A\partial^\mu B - (\partial^\mu A)B. \tag{294}$$

The conserved charge is

$$Q = \int d^3x\, j^0 = -i\int d^3x\, \phi^\dagger\overleftrightarrow{\partial_0}\phi \tag{295}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\vec{p}}a_{\vec{p}} - b^\dagger_{\vec{p}}b_{\vec{p}}\right). \tag{296}$$

The last equality follows by plugging in the expression for the field in terms of two sets of creation operators from the exercises. We see that the states created by a/b are particles/antiparticles: they have the same mass, but opposite charge.

6.1 Local symmetry and gauge fields

The symmetry transformation $\phi(x) \to e^{i\alpha}\phi(x)$ is called *global*, because α is the same for *all* spacetime points. This does not create any problems with causality, *because* it is a symmetry transformation, i.e., physical results are unchanged. Nevertheless, it seems more in the "spirit" of special relativity to make the symmetry transformation *local*, i.e., dependent on the spacetime point, $\phi(x) \to e^{i\alpha(x)}\phi(x)$.
It is one of the profound results of theoretical physics that this requires the introduction of a *gauge field*, which is a massless spin-1-field. In the case that the conserved global charge is interpreted as electric charge, the corresponding field is the gauge field of electromagnetism and its quantized excitations are the photons.

In this chapter, we will learn about the connection between local symmetries and gauge fields and we will consider their quantization in the canonical and the path-integral framework.

Mini-Exercise 14. How does $\partial_\mu\phi^\dagger\partial^\mu\phi$ transform under $\phi \to e^{i\alpha(x)}\phi$, $\phi^\dagger \to e^{-i\alpha(x)}\phi^\dagger$? We need this to figure out how $\mathcal{L}$ transforms.

Solution.

$$\partial_\mu\Big(e^{-i\alpha(x)}\phi^\dagger(x)\Big)\partial^\mu\Big(\phi(x)e^{i\alpha(x)}\Big) = \big(-i(\partial_\mu\alpha)\phi^\dagger + \partial_\mu\phi^\dagger\big)((\partial^\mu\phi) + i\phi\partial^\mu\alpha).$$

A mass term, $m^2\phi^\dagger\phi$, or, more generally, a globally $U(1)$-symmetric potential $V(\phi) = V(|\phi^\dagger\phi|)$, which can only depend on $\phi^\dagger\phi$, is automatically invariant under the local symmetry as well. This is because the potential itself is ultralocal, i.e., only depends on a single spacetime point. In contrast, the kinetic term is not invariant, because it depends on spacetime points which lie an infinitesimal distance apart.

Now we remember from the relativistic formulation of electrodynamics, that the gauge field $A_\mu(x)$ transforms according to

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\Lambda(x) \tag{297}$$

under a gauge transformation. $\Lambda(x)$ is a function that we can choose freely; under any choice of $\Lambda(x)$, the physics stays invariant. We can use this to define a *covariant derivative*

$$D_\mu = \partial_\mu + iA_\mu \tag{298}$$

and choose $\Lambda(x) = \alpha(x)$. Then we have that $D_\mu\phi$ transforms just by a phase $e^{i\alpha(x)}$ under the $U(1)$ symmetry:

$$\begin{aligned} D_\mu\phi \to D_\mu{}'\phi' &= \big(\partial_\mu + iA_\mu{}'\big)e^{i\alpha(x)}\phi(x) && (299)\\ &= (\partial_\mu + iA_\mu - i\partial_\mu\alpha(x))e^{i\alpha(x)}\phi(x) && (300)\\ &= e^{i\alpha(x)}(\cancel{\partial_\mu i\alpha(x)} + \partial_\mu + iA_\mu - \cancel{i\partial_\mu\alpha(x)})\phi(x) && (301)\\ &= e^{i\alpha(x)}D_\mu\phi. && (302) \end{aligned}$$

Thus we are automatically led to the introduction of *gauge fields*, if we promote global symmetries to be local. Thus, each local symmetry results in an *interaction* that is mediated by the gauge field.

We will later (in QFT II), generalize this to non-Abelian symmetry groups, e.g., $SU(2)$ and $SU(3)$, for which the gauge bosons interact with each other and which are part of the Standard Model.

Let us dive a little further into the mathematics of gauge fields before we quantize. Why do we actually need the gauge connection (which is just a more mathematical term for the gauge field)? The answer is that it is necessary to evaluate the change of the field from one spacetime point to another, i.e., to define a derivative. The definition that is fine for a field without a local symmetry transformation,

$$n^\mu\partial_\mu\phi(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(\phi(x+\varepsilon n) - \phi(x)) \qquad \text{with unit vector } n^\mu \tag{303}$$

does not work, i.e., the partial derivative fails, because, for a local symmetry, the phases of $\phi(x)$ and $\phi(x+\varepsilon n)$ are different.

Instead, we have to *parallel transport* $\phi(x)$ from one point to a neighboring point in such a way that the phase difference between the field at both points is removed. $U(y,x)\phi(x)$ fulfills the

requirement of transforming just like $\phi(y)$, if

$$U'(y,x) = e^{i\alpha(y)} U(y,x) e^{-i\alpha(x)}. \tag{304}$$

Therefore, we can define a covariant derivative D_μ, such that

$$n^\mu D_\mu \phi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\phi(x + \varepsilon n) - U(x + \varepsilon n, x)\phi(x)). \tag{305}$$

How does this relate to our previous definition? We assume that $U(x,x) = \mathbb{1}$, and $U(x,y)$ to be smooth, so that we can Taylor expand

$$U(x + \varepsilon n, x) = 1 - \underbrace{i\,\varepsilon\, n^\mu A_\mu(x)}_{\substack{\text{this is the } \textit{linear} \\ \text{Taylor coefficient and} \\ \text{this definition of} \\ A_\mu \text{ and } D_\mu \text{agrees} \\ \text{with the earlier one.}}} + \dots \tag{306}$$

From $U(x,y)$, we can define an object that is conceptually interesting and also very useful in QFT, because it forms the basis of so-called lattice simulations, i.e., numerical simulations of QFT, where spacetime is discretized into a lattice. This object is a *Wilson line*

$$U(y,x,C) = e^{i \int_{C(x,y)} dx^\mu \, A_\mu}, \tag{307}$$

where $C(x,y)$ is a curve from x to y.
From U, the name *gauge connection* for A_μ makes more sense: $U(x+\varepsilon n, x)$ is the phase associated to the straight line from x to $x + \varepsilon n$, so A_μ *connects* different points in a gauge-invariant way.

6.2 Dynamics for A^μ

We have introduced a new field, A^μ, so we need to specify its dynamics. Because $A_\mu \to A_\mu - \partial_\mu \alpha$ under a gauge transformation, $(\partial_\mu A^\mu)(\partial_\nu A^\nu)$ is *not* gauge invariant, but $F_{\mu\nu}F^{\mu\nu}$ is, because $F_{\mu\nu}$ is gauge invariant. Here, we are relying on using knowledge from classical electrodynamics.

6.3 Quantization of the gauge field

We focus on the free (non-interacting) theory first, so our Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

We try to treat each of the 4 components of A^μ as an independent field, finding the associated canonically conjugate momentum field.

Mini-Exercise 15. Find $\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu}$.

Solution.

$$\begin{aligned}\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} &= \frac{\partial}{\partial(\partial_0 A_\mu)}\left(-\frac{1}{4}F_{\rho\nu}F_{\sigma\tau}\eta^{\rho\sigma}\eta^{\nu\tau}\right) \\ &= -\frac{1}{2}F_{\rho\nu}\eta^{\rho\sigma}\eta^{\nu\tau}\frac{\partial}{\partial(\partial_0 A_\mu)}(\partial_0 A_\tau - \partial_\tau A_0) \\ &= -\frac{1}{2}F_{\rho\nu}\left(\eta^{\rho 0}\eta^{\nu\mu} - \eta^{\rho\mu}\eta^{\nu 0}\right) = F^{\mu 0}.\end{aligned}$$

This is a problem, because $F^{00} = 0$ and thus $\Pi^0 = 0$. Therefore, the quantization deals *not* with 4 independent fields, but rather with a *constraint*. That such a constraint is there, makes sense, because the gauge symmetry means that not all four components of A^μ are physical and independent; the gauge symmetry makes some of the components unphysical.

There are several ways to deal with this:

i) Choose radiation gauge ($\vec{\nabla}\cdot\vec{A} = 0,\ A^0 = 0$), then quantize.

ii) Gupta-Bleuler quantization: impose the gauge condition on the Hilbert space.

iii) Hamiltonian quantization with constraints à la Dirac.

iv) Faddeev-Popov trick in the path-integral.

We will follow method i) in an exercise. In this procedure, only the 2 physical polarizations of the photon are quantized, which is an advantage. As a disadvantage, we give up manifest Lorentz invariance, because the radiation gauge condition $\vec{\nabla}\cdot\vec{A} = 0$ and $A^0 = 0$ is not manifestly Lorentz invariant.

Method ii) has the advantage that we can work in a gauge condition that is manifestly invariant, $\partial_\mu A^\mu = 0$. We will also see directly the connection between gauge-symmetry and unitarity (i.e., positivity of probabilities), because we will encounter negative-norm states which are *unphysical* and removed by the gauge condition.

Method iii) is generalizable to other systems with constraints and is therefore generalized in Loop Quantum Gravity. We will not cover it here, but you can read up on it in the book by Fradkin.

Method iv) generalized to non-Abelian symmetry groups, e.g., $SU(2)$ and $SU(3)$ and is therefore used in quantization of the Standard Model.

6.4 Gupta-Bleuler quantization

In order to choose a gauge condition that is compatible with Lorentz-invariance, we cannot single out any given component of A^μ. Thus, we choose Lorenz gauge[11]

$$\partial_\mu A^\mu = 0. \tag{308}$$

[11] This gauge is named after the physicist Lorenz, who is a different person from the Lorentz after whom Lorentz transformations are named. Lorentz was Dutch, Lorenz was Danish.

The gauge-fixed Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2. \tag{309}$$

λ is called the gauge parameter, and we will choose $\lambda = 1$ (Feynman gauge), which is a special case of the family of gauges that constitute Lorenz gauge. The gauge-fixing condition is added quadratically, so that it shows up as part of the kinetic term for A^μ and thus affects the propagation of the four components of A^μ. In particular, in this way, there will be a contribution $\sim \partial_\mu A^\mu$ in the canonically conjugate momentum field Π^μ.

Then,

$$\Pi^\mu = F^{\mu 0} - \eta^{\mu 0}\partial_\chi A^\chi, \tag{310}$$

$$\text{and } \mathcal{L} = -\frac{1}{2}\partial_\mu A^\nu \partial^\mu A_\nu + \text{total derivative}. \tag{311}$$

This looks like the Lagrangian for four massless, real scalar fields (the 4 components of A^μ). However, the sign of the kinetic term of A_0 is wrong. In the exercises, you will see that this problem persists for a massive vector field, so that the Proca action actually contains the constraint that A^0 does not propagate.
We also note that if we impose Lorenz gauge $\partial_\mu A^\mu = 0$ at this point, then we get back the result $\Pi^0 = 0$. The solution is that we *first quantize and then impose the gauge-condition* $\partial_\mu A^\mu = 0$ on the Hilbert space, i.e., we will define a physical subspace F_{physical}, such that

$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = 0 \qquad \text{for } |\psi\rangle \in F_{\text{physical}}. \tag{312}$$

Let's see how this construction works. We quantize by demanding

$$[A_\mu, A_\nu] = 0 = [\Pi_\mu, \Pi_\nu], \tag{313}$$

$$[A_\mu(\vec{x}), \Pi^\nu(\vec{y})] = i\,\eta_\mu{}^\nu \delta^3(\vec{x}-\vec{y}) = i\,\delta_\mu{}^\nu \delta^3(\vec{x}-\vec{y}). \tag{314}$$

Now we Fourier transform, introduce a and $a^\dagger$ and go over to Heisenberg fields:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{2\omega_{\vec{k}}}}\Big(a_{\vec{k},\mu}e^{-ikx} + a^\dagger_{\vec{k},\mu}e^{ikx}\Big). \tag{315}$$

Accordingly, we arrive at

$$\Big[a_{\vec{k},\mu}\,, a^\dagger_{\vec{k}',\nu}\Big] = -\eta_{\mu\nu} 2\,\underbrace{\omega_{\vec{k}}}_{\shortparallel\; k^0}\,(2\pi)^3\delta^3(\vec{k}-\vec{k}'). \tag{316}$$

Comparing to the case of the scalar field, $\Big[a_{\vec{p}}, a^\dagger_{\vec{q}}\Big] \sim \delta^3(\vec{p}-\vec{q})$, we note that the sign of $\Big[a_0, a_0^\dagger\Big]$ is wrong. That there is a problem with the 0-component of A_μ is not too surprising, because this is the component for which the canonically conjugate momentum field vanishes, if the gauge condition is imposed. Now, we can see more precisely what the problem is, when we try to define the Hilbert space (Fock space):

1) The metric on the Hilbert space is not positive definite, i.e., some states have *negative* norm.

$$\left\| a_0^\dagger |0\rangle \right\|^2 = \langle 0 | a_0 \, a_0^\dagger | 0 \rangle = \langle 0 | \Big(a_0^\dagger \, a_0 - 1 \Big) | 0 \rangle = -1. \tag{317}$$

Because of the wrong sign of the commutator, this is -1, not $+1$. We might try to solve this problem by switching the roles of a_μ and $a_\mu^\dagger$, but then the problem simply reappears for a_i, so that is not a solution.

2) Our choice of gauge, $\partial_\mu A^\mu = 0$, *cannot* be imposed at the operator level, because $[A_0, \partial_\mu A^\mu] = 0$ would have to hold for that. However, $\partial_\mu A^\mu = \Pi_0$, and $[A_0, \Pi_0] \neq 0$, thus imposing the gauge condition on the operator level is incompatible with the commutation relations.

Gupta and Bleuler's solution to both problems is to define a physical subspace F_{subspace} of the Fock space F by

$$\langle \psi | \partial_\mu A^\mu | \psi \rangle = 0 \qquad \forall \ |\psi\rangle \in F_{\text{physical}}, \tag{318}$$

i.e., physical states are annihilated by the annihilation part of $\partial_\mu A^\mu$:

$$\partial_\mu A^\mu \Big|_{\text{annihilation}} |\psi\rangle = 0, \tag{319}$$

because the part of A^μ containing creation operators would not vanish on the vacuum state and thus the vacuum state would be declared to not be part of the physical Hilbert space; thus that condition would be too severe. F_{physical} contains no negative norm states, as we will see, but it still contains zero-norm states.
We define $F_0 \coloneqq \{|\psi\rangle \in F_{\text{physical}} : \||\psi\rangle\| = 0\}$, so that we can define the physical Hilbert space as the quotient $\mathcal{H} = F_{\text{physical}}/F_0$.
This means that $\mathcal{H}$ is the space of equivalence classes of vectors from $\mathcal{H}$, with the equivalence relation $\sim$ defined as

$$|\psi\rangle \sim |\psi'\rangle \iff \| |\psi\rangle - |\psi'\rangle \| = 0, \tag{320}$$

i.e., $|\psi\rangle$ and $|\psi'\rangle$ differ by the addition of a state of zero norm. To work out the physical implications, we will work in terms of the polarization vectors.

6.4.1 Polarization vectors

A general 1-photon state is a linear combination of states $a_{\vec{k},\mu}^\dagger |0\rangle$, that can be written as

$$-\epsilon^\mu(k) \, a_{\vec{k},\mu}^\dagger |0\rangle, \tag{321}$$

with fixed $\vec{k}$; where the negative sign is there to avoid a minus from the metric when lowering the index and considering spacelike polarization. There are 4 independent polarizations $\epsilon^\mu(k)$ for any $\vec{k}$ and different ways to choose a basis.
The basis we will use is based on a light-like auxiliary vector n, $n^2 = 0$, with $n \nparallel k$ (n not parallel

to k), such that we can go to a coordinate system in which

$$n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \qquad k = \left|\vec{k}\right| \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{322}$$

Then we can choose the 4 polarization basis vectors as

$$\epsilon^u = n, \tag{323}$$

$$\epsilon^L \sim k, \tag{324}$$

$$\epsilon^{(1)}, \epsilon^{(2)} \text{ orthogonal to } n, k. \tag{325}$$

The meaning of "u" and "L" will become clear below. These conditions are fulfilled by

$$\epsilon^u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \epsilon^L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{326}$$

A general 1-photon state is

$$|\epsilon, k\rangle = -\epsilon^\mu(k) a^\dagger_{\vec{k},\mu} |0\rangle, \tag{327}$$

and the scalar product of two 1-particle states is

$$\left\langle \epsilon', k' \middle| \epsilon, k \right\rangle = \epsilon'^\mu(k')^* \epsilon^\nu(k) \langle 0 | a_{\vec{k}',\mu}\, a^\dagger_{\vec{k},\nu} | 0 \rangle \tag{328}$$

$$= -\big(\epsilon' \cdot \epsilon\big) 2k^0 (2\pi)^3 \delta^3(\vec{k} - \vec{k}'). \tag{329}$$

Based on this, we can now construct the Hilbert space in terms of polarizations. Due to our discussion of the little group, we already know that only two polarization vectors will play a role in physical states of the photon. The condition

$$\partial_\mu A^\mu \Big|_{\text{annihilation}} |\psi\rangle = 0 \tag{330}$$

in terms of polarization vectors is the subject of the next mini exercise:

Mini-Exercise 16. What does $\partial_\mu A^\mu\big|_{\text{annihilation}} |\psi\rangle = 0$ translate into in terms of the polarization vectors?

Solution.

$$\begin{aligned}
\partial_\mu A^\mu\Big|_{\text{annihilation}} |\epsilon, q\rangle &= 0 \\
k_\mu a^\mu_{\vec{k}} |\epsilon, q\rangle &= 0 \\
k_\mu \underbrace{a^\mu_{\vec{k}} a^{\nu\dagger}_{\vec{q}}}_{\sim \eta^{\mu\nu}\delta^3(\vec{k}-\vec{q})} \epsilon_\nu(q) |0\rangle &= 0 \\
k_\mu \epsilon^\mu(k) &= 0.
\end{aligned}$$

This condition holds automatically for $\epsilon^{(1)}$, $\epsilon^{(2)}$, ϵ^L, but is violated for ϵ^u, i.e., ϵ^u is the *unphysical* polarization. Therefore, it makes sense to perform a linear transformation on the space of creation/annihilation operators:

$$\alpha^\dagger_{\vec{k},(u,L,1,2)} = \epsilon^{\mu\,(u,L,1,2)}(k) a^\dagger_{\vec{k},\mu}. \tag{331}$$

6.4.2 The Fock space

F can be built with the $a^\dagger$ or the $\alpha^\dagger$, but we can now define F_{physical} as the space built by $\alpha^\dagger_{\vec{k},(L,1,2)}$, because for all those states

$$(k \cdot a_{\vec{k}})(\text{products of } \epsilon^{(L,1,2)}_{\vec{q}} \cdot a^\dagger_{\vec{q}}) |0\rangle = 0. \tag{332}$$

Thus,

$$|\psi\rangle \in F_{\text{physical}} \to |\psi\rangle = (\text{products of } \alpha^\dagger_{(L,1,2)}) |0\rangle. \tag{333}$$

Next, we can isolate the states with zero norm:
Since $\epsilon_L^2 = 0$, $\||\psi\rangle\| = 0$ if and only if at least one $\alpha^\dagger_L$ appears in $|\psi\rangle$. Such zero-norm states do not change the expectation values of observables:

$$\langle\psi'|\mathcal{O}|\psi'\rangle = \langle\psi|\mathcal{O}|\psi\rangle, \qquad \text{if } |\psi'\rangle = |\psi\rangle + (\ldots \alpha^\dagger_L \ldots)|0\rangle. \tag{334}$$

We demonstrate this claim for the Hamiltonian:

$$\begin{aligned}
H &= \int d^3x \left(\Pi^\mu \dot{A}_\mu - \mathcal{L}\right) && (335)\\
&= \int \frac{d^3k}{(2\pi)^3}\, k_0 \left(-a^\dagger_{\vec{k},\mu} a^\mu_{\vec{k}}\right) && (336)\\
&= \int \frac{d^3k}{(2\pi)^3}\, k_0 \left(\sum_{i=1}^{2} \alpha^\dagger_{\vec{k},i} \alpha_{\vec{k},i} - \underbrace{\left[\alpha^\dagger_{\vec{k},u} \alpha_{\vec{k},L} + \alpha^\dagger_{\vec{k},L} \alpha_{\vec{k},u}\right]}_{\substack{\text{vanishes inside} \\ \text{a ``physical'' state}}} \right). && (337)
\end{aligned}$$

Ultimately, the two transverse polarizations $\epsilon^{(1,2)}$ are related to physical states; just as they already are in the classical theory.

Note: From experiment, we know that there are 2 physical degrees of freedom of the photon, i.e., just two independent polarizations exist. Thus, it makes sense that one polarization is unphysical, but we would actually expect two. The reason that another polarization is not removed by the

gauge condition, but instead has zero norm and therefore does not show up in expectation values of physical states, is the *residual gauge freedom*.
In the classical theory, $\partial_\mu A^\mu = 0$ does not fix the gauge freedom completely, $A_\mu \to A_\mu + \partial_\mu \chi$ still has some choices of χ. This is easiest to see in Fourier space, where $\tilde{A}_\mu \to \tilde{A}_\mu + i k_\mu \tilde{\chi}$ is the gauge freedom.
If k_μ is lightlike, adding a term $\sim k_\mu$ does not destroy the gauge condition, because $k^2 = 0$; the new field still obeys $k_\mu \tilde{A}^\mu = 0$. The quantum analogue of this is to add states involving longitudinal polarizations.

General comment on gauge symmetries: they are not so much physical symmetries, but redundancies of description, because the physical state is the Hilbert space, in which there are no gauge transformations, because we've first gone to F_{physical} and then moded out F_0.

6.5 Path-integral quantization of gauge theories

(Fradkin, Srednicki ...)

6.5.1 Gauge symmetry = gauge redundancy

There is an important conceptual difference between a global and a local symmetry: A global symmetry in a QFT is a genuine symmetry between physically viable configurations; it implies that there are quantities (most importantly the action) that are equal for the different configurations related to each other by the symmetry. For instance, for an O(2) symmetry that rotates two scalar fields $\phi_1(x), \phi_2(x)$ into each other, the action is the same, irrespective of whether field configuration is, e.g., $\{\phi_1(x) = \sin(x),\ \phi_2(x) = 0\}$ or $\{\phi_1(x) = 0,\ \phi_2(x) = \sin(x)\}$. Nevertheless, the two different configurations are physically distinct (and can be distinguished by measurements) and the system has two physical degrees of freedom (one linked to ϕ_1 and one to ϕ_2).
For a local symmetry, this is different, because the gauge transformation relates different configurations of $A_\mu(x)$ to each other in a way that is *not accessible to any measurement; the configurations* $A_\mu(x)$ *and* $A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x)$ *are physically indistinguishable.* Thus, gauge symmetry is not really a symmetry, but rather a *redundancy of description.* We see this from the fact that not all four components of A_μ give rise to physical degrees of freedom; despite the fact that $A_\mu(x)$ has four components; the photon only has two degrees of freedom.
Gauge theories can be written in terms of gauge-invariant (i.e., physical) quantities, but at the cost of giving up locality. These are the Wilson-loops (briefly introduced above). They form the basis of lattice simulations of gauge theories, but outside of numerical simulations, formulating a QFT in terms of these non-local objects is difficult and rarely done in practise. Typically, given the choice between the gauge redundancy and non-locality, one works in the gauge-redundant, local description in terms of the gauge field.

6.5.2 The path integral for photons: kinetic term as projector

(Literature for this subsection: Srednicki).
The path integral for photons is

$$Z[J] = \int \mathcal{D}A\, e^{iS[A] + i\int d^4x J^\mu A_\mu}. \tag{338}$$

The currents that photons couple to are the conserved currents for scalar (and later also spin 1/2) fields that follow from Noether's theorem for a local U(1) symmetry, i.e.,

$$\partial_\mu J^\mu = 0. \tag{339}$$

We write out the action and the current-terms in the exponential in Eq. (338) in Fourier space:

$$\begin{aligned} S &= -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu} + \int d^4x J_\mu A^\mu & (340)\\ &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\left(-\tilde{A}_\mu(k)\left(k^2\eta^{\mu\nu}-k^\mu k^\nu\right)\tilde{A}_\nu(-k) + \tilde{J}^\mu(k)\tilde{A}_\mu(-k) + \tilde{J}^\mu(-k)\tilde{A}_\mu(k)\right). & (341)\end{aligned}$$

Now we are interested in performing the path-integral over the gauge field. Because the gauge field couples to the complex scalar through the current, we can already see that, just like when we integrated over quantum fluctuations of the scalar field, the integral over quantum fluctuations of the gauge field will generate new interactions for the scalar field. However, there is an apparent problem in performing the path integral over the gauge field, and we will see (and then solve) it now.

The path integral is quadratic in A_μ. Thus it is a Gaussian integral and we can perform it. Because the exponential contains a quadratic and a linear term, we need to shift $\tilde{A}_\mu$ to "complete the square" according to the path-integral analogue of

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, e^{i\,x^2 a + ib\,x} &= \int_{-\infty}^{\infty} dy\, e^{i\left(y-\frac{b}{2a}\right)^2 a + ib\left(y-\frac{b}{2a}\right)} \quad \text{where } y = x + \frac{b}{2a}\\ &= \int_{-\infty}^{\infty} dy\, e^{ia\,y^2 - i\frac{b^2}{4a}}. \end{aligned} \tag{342}$$

In our case, this means that we need to shift

$$\tilde{A}_\mu(k) \to A_\mu - \frac{1}{2}J^\nu\left(k^2\eta_{\mu\nu} - k_\mu k_\nu\right)^{-1}, \tag{343}$$

i.e., we need to invert the matrix

$$k^2 P^{\mu\nu}(k) := k^2\eta^{\mu\nu} - k^\mu k^\nu. \tag{344}$$

This matrix provides the kinetic term for the gauge field. It turns out, that it is not invertible, because is has a vanishing eigenvalues. To see this, do the following

Mini-Exercise 17. What is $P^{\mu\nu}k_\nu$?

Solution.

$$P^{\mu\nu}k_\nu = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)k_\nu = k^\mu - \frac{k^\mu\, k^2}{k^2} = 0.$$

Thus, k^ν is an eigenvector of $P^{\mu\nu}$, with eigenvalue zero. This is related to the fact that the polarization vector $\epsilon^L \sim k$ does not describe a physical polarization. In fact, $P^{\mu\nu}$ is a projection matrix. A projection matrix is defined by the property $P^2 = P$.

Mini-Exercise 18. Check that $P^{\mu\nu}(k)P_\nu{}^\lambda(k) = P^{\mu\lambda}(k)$.

Solution.

$$P^{\mu\nu}(k)P_\nu{}^\lambda(k) = \left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\cdot\left(\eta^\lambda_\nu - \frac{k_\nu k^\lambda}{k^2}\right)$$
$$= \eta^{\mu\lambda} - \frac{k^\mu k^\lambda}{k^2} - \frac{k^\mu k^\lambda}{k^2} + \frac{k^\mu\, k^2\, k^\lambda}{(k^2)^2} = \eta^{\mu\lambda} - \frac{k^\mu k^\lambda}{k^2} = P^{\mu\lambda}(k).$$

For any projection matrices, the eigenvalues are either zero or one; in the present case, one of the eigenvalues is zero. Thus, it seems that we are prevented from carrying out the path integral for the gauge field by the projector-properties of the kinetic term of the gauge field. This is a consequence of the gauge symmetry, due to which not all components of A^μ are physical and not all propagate as physical photons.

However, it turns out that the path integral becomes doable because the longitudinal polarization drops out entirely. To see this, we imagine writing $\tilde{A}_\mu(k)$ as a sum of contributions proportional to each of the polarization vectors. Then the component $\sim \epsilon^L$ drops out of the kinetic term (it is "projected out"). In addition, it also drops out of the product $J^\mu A_\mu$, because

$$\epsilon^L_\mu \cdot \tilde{J}^\mu(k) \sim k_\mu \tilde{J}^\mu(k) = 0. \tag{345}$$

Here, we first used that $\epsilon^L \sim k$ and then we used that the current is conserved,

$$\partial_\mu J^\mu \Rightarrow k_\mu \tilde{J}^\mu(k). \tag{346}$$

Thus, the integration over ϵ^L is trivial, because ϵ^L does not show up in the exponential. Therefore, it just gives us a constant prefactor of the path integral, which we can equally well just drop from the outset, because it just changes the overall normalization. This is enough to make the path integral for photons calculable.

Nevertheless, it is very instructive to think about the path integral for gauge theories in a more abstract way. First of all, it helps us to understand the properties of gauge symmetries more deeply; second, what we will do now generalizes to non-Abelian symmetry groups (whereas the simple dropping-out-of-the-path-integral does not generalize to non-Abelian symmetries).

6.5.3 Conceptual aspects: gauge orbits and gauge fixing

$$Z[J] = \int \mathcal{D}A_\mu\, e^{i\int d^4x\, \mathcal{L}} \qquad \text{with } \mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu \tag{347}$$

is the straightforward generalization of the path-integral expression for the scalar field. However, the expression is not well-defined, because field configurations (sometimes also referred to as "histories", because the value of the field at each point in space is give for all times) that differ only by a gauge transformation, and are thus physically equivalent, are included. Because $\mathcal{D}A_\mu$ and S are gauge invariant, the weight of such configurations is equal. Therefore, $Z[J]$ has a divergence of the form

$$V(G)^{\mathcal{V}}, \tag{348}$$

where $V(G)$ is the volume of the gauge group (2π for $U(1)$) and $\mathcal{V}$ the spacetime volume. We will

now implement a procedure that will straightforwardly generalize to non-Abelian gauge symmetries and thus to the Standard Model. We will then see, that for the case of $U(1)$, i.e., photons, the path integral simplifies, but we will still use $U(1)$ as our example for the more general path-integral quantization of gauge theories.

To do so, it is useful to introduce a few more concepts:
In general, the vector potential A_μ transforms in the *adjoint* representation of the gauge group, which means that it can be written as a linear combination of the group generators, i.e., it takes values in the algebra of the group. A field in the adjoint representation of a Lie group transforms as

$$A_\mu \to A'_\mu = A^U_\mu = U A_\mu U^\dagger + iU\partial_\mu U^\dagger, \tag{349}$$

which holds for $U(1)$, but also $SU(2)$ and $SU(3)$ etc. For $U(1)$, we have that $U = e^{i\alpha(x)}$, such that

Mini-Exercise 19. What does A^U_μ look like?

Solution.

$$\begin{aligned} A^U_\mu &= e^{i\alpha(x)} A_\mu e^{-i\alpha(x)} + ie^{i\alpha(x)}(\partial_\mu(-i)\alpha(x))e^{-i\alpha(x)} \\ &= A_\mu + \partial_\mu \alpha(x). \end{aligned}$$

This is the more familiar form of the gauge transformation that we already know. We will work with the more abstract, general form in Eq. (349), because then our results apply to other gauge groups directly.

To properly understand the effect of a gauge symmetry (which is really a redundancy of description), we will consider the structure of the *configuration space* for the gauge field. The configuration space is an (infinite-dimensional) space; each point in it denotes a field configuration $A_\mu(x)$. In other words, the configuration space contains all possible configurations of the fields, i.e., all possible sets of four functions of the spacetime coordinates that make up the four-vector function $A_\mu(x)$. Note that the configuration space is *not* just those configurations $A_\mu(x)$ which satisfy the equations of motion. The configuration space is the space that we are integrating over in the path integral.
It is useful to think of the configuration space of A_μ as follows:
The configuration space consists of *classes* of gauge configurations. Within each class, all field configurations are physically equivalent and thus related to each other by a gauge transformation, cf. left panel in Fig. 8. We would like to *pick one representative* from each class to get rid of the factor $V(G)^{\mathcal{V}}$ in front of the path integral. This is achieved by a gauge condition, cf. right panel in Fig. 8.

Mini-Exercise. This is all rather abstract. Team up in groups of 2-3 to discuss these ideas.

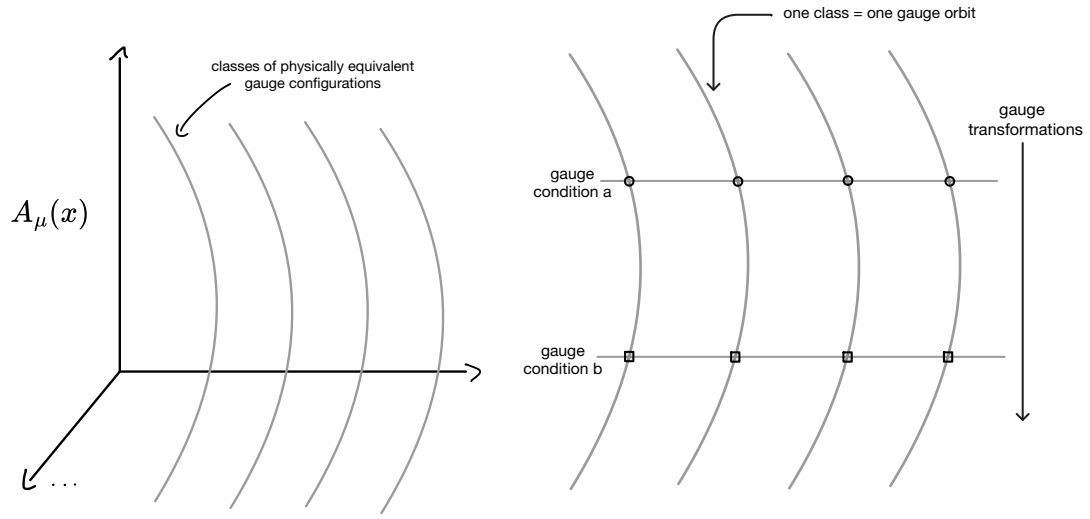


Figure 8: Left panel: One can move "along" a class by infinitesimally changing $\alpha(x)$ in a suitable way (this is also sometimes called a "gauge orbit". The drawn space of $A_\mu(x)$ is infinite-dimensional; each point corresponds to a distinct field configuration; the axis are labeled by some suitable choice of basis functions.
Right panel: ○: representative field configuration from each class, picked by gauge condition a.
□: representative field configuration from each class, picked by gauge condition b.
○ and □ along one orbit are physically equivalent, but mathematically distinct field configurations.

6.5.4 Fadeev-Popov-trick

How to achieve this selection?
Naively, we would think that we should write

$$Z[J] \sim \int \mathcal{D}A_\mu \, \delta(\text{gauge condition}) e^{iS}. \tag{350}$$

However, in this way we would explicitly break the gauge symmetry through our choice of gauge condition. This must not be the case, because then we get problems with unitarity, because the modes that come with negative norm in F are only removed because of gauge symmetry. Instead, we will use the functional version of

$$1 = \int dx \, \delta(f(x)) \sum_i \left(\frac{\mathrm{d}f}{\mathrm{d}x} \right) \Big|_{x=x_i}, \tag{351}$$

where x_i are the zeros of $f(x)$. Note that the right-hand-side of the expression contains the desired delta function. However, the left-hand side is 1, due to the Jacobian factor on the right-hand side. Thus, if we use an expression like this, we are inserting a 1, but are at the same time achieving our goal to have a delta-function of the gauge condition.
Under the generalization from an integral over x to an integral over all possible gauge transforma-

tions, it similarly holds that

$$1 = \underbrace{\int \mathcal{D}U}_{\substack{\text{integral over}\\ \text{all gauge}\\ \text{transformations}}} \quad \underbrace{\delta(g(A_\mu^U)}_{\substack{\text{gauge condition}\\ g(A_\mu^U)=0}} \quad \underbrace{\det \left|\frac{\delta g}{\delta U}\right|_{g=0}}_{\text{Jacobian}}. \tag{352}$$

We assume that g only has one zero.[12] Thus, we can write:

$$Z[J] = \int \mathcal{D}A_\mu \, e^{iS[A]+i\int d^4x \, J_\mu A^\mu} \int \mathcal{D}U \, \delta\big(g(A_\mu^U)\big) \det \left|\frac{\delta g}{\delta U}\right|_{g=0}. \tag{353}$$

Let us take a closer look at $\det \left|\frac{\delta g}{\delta U}\right|_{g=0}$, for the case $g = \partial_\mu A^\mu$, i.e., Lorenz gauge. $\delta A^\mu = \partial_\mu \alpha(x)$, and thus

$$\frac{\delta g}{\delta U} = \frac{\delta g}{\delta A_\mu^U} \frac{\delta A_\mu^U}{\delta \alpha} \frac{\delta \alpha}{\delta U} = (\partial_\mu \partial^\mu) e^{i\alpha}. \tag{354}$$

The net result is

$$\det \left|\frac{\delta g}{\delta U}\right|_{g=0} = \det \partial^2. \tag{355}$$

This is independent of A_μ, and can just be factored out of the path integral. This will be different for non-Abelian gauge symmetries, where we will have to introduce the so-called Faddeev-Popov ghosts at this step (see QFT II).
Now we use that

i) $\det \left|\frac{\delta g}{\delta U}\right|_{g=0}$ is gauge invariant, i.e., it is the same for each gauge transformation (see above).

ii) $S[A_\mu]$ is gauge invariant, so $S[A_\mu] = S[A_\mu^U]$; and the coupling to the current is gauge invariant (because it arises precisely to make the charged field in the current have a gauge-invariant kinetic term).

iii) $\mathcal{D}A_\mu$ is gauge invariant, because we are integrating over *all* configurations, so $\mathcal{D}A_\mu = \mathcal{D}A_\mu^U$.

$$\implies Z[J] = \int \mathcal{D}U \int \mathcal{D}A_\mu^U \, e^{iS[A_\mu^U]+i\int d^4x \, J_\mu A^{U\,\mu}} \, \delta\big(g(A_\mu^U)\big) \det \left|\frac{\delta g}{\delta U}\right|_{g=0}. \tag{356}$$

Now A_μ^U appears everywhere in the integrand and we can simply relabel the integration variable, $A_\mu^U \to A_\mu$, so that

$$Z[J] = \int \mathcal{D}U \int \mathcal{D}A_\mu \, e^{iS[A]+i\int J\cdot A} \, \delta(g(A)) \det \left|\frac{\delta g}{\delta U}\right|_{g=0}. \tag{357}$$

$\int \mathcal{D}U$ is now a *prefactor* that factorizes. It is precisely the integral over the volume of the gauge group that we wanted to get rid of. We perform one more step to make this expression usable in practice:
We use that

$$\delta(g(A)) = \lim_{\xi\to 0} e^{\frac{i}{2\xi}\int d^4x \, (g(A))^2} \tag{358}$$

and then generalize to finite ξ. The limit $\xi \to 0$ imposes the gauge condition sharply. In this way, the gauge-fixing condition just appears alongside the action.

[12] In non-Abelian gauge theories, this assumption does not hold beyond perturbation theory and leads to the Gribov problem, that still remains unsolved and subject of research.

At the end, we obtain

$$Z[J] = \int \mathcal{D}A_\mu \, e^{iS[A]+i\int d^4x \, A_\mu J^\mu + \frac{i}{2\xi}\int d^4x (\partial_\mu A^\mu)^2}. \tag{359}$$

7 Spin 1/2 field

7.1 Gamma-matrices (Dirac matrices) and spinors

We saw in our study of the representation of the Lorentz group, that there are two representations with spin 1/2, each of which has two degrees of freedom. We already called them "left-handed spinor" and "right-handed spinor".
We will group them together into a reducible representation, which is a Dirac spinor. As we will discover, it will have 4 components, namely right-handed and left-handed particle and antiparticle. It turns out that a representation of the Lorentz algebra,

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}) \tag{360}$$

can be found, by defining

$$M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \tag{361}$$

where γ_μ is called a "gamma matrix" or "Dirac matrix" and the defining property of γ-matrices is that they satisfy the

$$\underline{\text{Clifford algebra:}} \qquad \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}. \tag{362}$$

Here, the γ's must clearly be matrices, so that their commutator, used to define M, does not simply vanish. The $\mathbb{1}$ in the Clifford algebra is a $\mathbb{1}$ in the indices of these matrices that we often suppress. One can show that the smallest rank for which the Clifford algebra can be satisfied is 4. For instance, we can define

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{363}$$

(where the 0's denote the $2x2$ submatrices, filled with zeros, and the $\mathbb{1}$'s are unity in $2x2$ matrices) and

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3 \tag{364}$$

with the *Pauli matrices*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{365}$$

Then,

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}. \tag{366}$$

It is not really surprising that we see the Pauli matrices appear (and appear twice in each γ). After all, we saw that there are 2 spin-1/2-representations of the Lorentz group, which each correspond to a fundamental representation of an $SU(2)$ (which is given by Pauli matrices). We're currently building a *reducible* representation that contains both of these irreducible representations together and thus has 2 sets of Pauli matrices. If we want to write this in a more compact way, we can define a 4-vector, where each of the components is a matrix,

$$\sigma^\mu = \begin{pmatrix} \mathbb{1} \\ \sigma^i \end{pmatrix} \quad \text{and another 4-vector} \quad \bar{\sigma}^\mu = \begin{pmatrix} \mathbb{1} \\ -\sigma^i \end{pmatrix}. \tag{367}$$

This allows us to write

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{368}$$

(Note: if you work in $(-,+,+,+)$ conventions, then γ^μ has an extra prefactor $(-i)$).

Mini-Exercise 20. Given $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, check that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}, \quad \text{for } (\mu,\nu) = (0,0) \text{ and } (\mu,\nu) = (i,j). \tag{369}$$

Solution.

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \\
&= \left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \right] \\
&= \left[\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu + & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu \end{pmatrix} + \begin{pmatrix} \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \right].
\end{aligned}$$

For $(\mu,\nu) = (0,0)$, we obtain

$$\begin{aligned}
\{\gamma^\mu, \gamma^\nu\} &= \left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \right] \\
&= +2 \cdot \mathbb{1} = +2\eta^{00}\mathbb{1}.
\end{aligned}$$

For $(\mu,\nu) = (i,j)$, we obtain

$$\begin{aligned}
\{\gamma^i, \gamma^j\} &= \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix} \\
&= -\begin{pmatrix} (\delta^{ij} + i\epsilon^{ijk}\sigma^k) & 0 \\ 0 & (\delta^{ij} + i\epsilon^{ijk}\sigma^k) \end{pmatrix} - \begin{pmatrix} (\delta^{ji} + i\epsilon^{jik}\sigma^k) & 0 \\ 0 & (\delta^{ji} + i\epsilon^{jik}\sigma^k) \end{pmatrix} \\
&= -2\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\delta^{ij} = 2\,\eta^{ij}\,\mathbb{1},
\end{aligned}$$

where we used that $\epsilon^{jik} = -\epsilon^{ijk}$.

We can also define a non-zero matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{370}$$

From the expression on the rhs in terms of $\epsilon_{\mu\nu\rho\sigma}$, it follows that

$$\{\gamma^5, \gamma^\mu\} = 0. \tag{371}$$

It also holds that $\left(\gamma^5\right)^2 = \mathbb{1}$, and, most importantly,

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{372}$$

We already see that γ^5 may later be helpful to isolate the two irreducible spin-1/2 representations in our reducible one, because we can define a set of *projection matrices*

$$P_L = \frac{1}{2}\left(\mathbb{1} - \gamma^5\right) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \tag{373}$$

$$\text{and } P_R = \frac{1}{2}\left(\mathbb{1} + \gamma^5\right) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{374}$$

The explicit expressions for the γ's (and γ^5) are in the *Weyl representation*. (It turns out, as we will see later, that we can perform linear transformations on the γ-matrices, $\gamma^\mu \to S\gamma^\mu S^{-1}$ and thereby write different representations.)

Now we must define the *field* that constitutes the spin-1/2-representation. We will call it a *spinor*. From our considerations, we know that

- it must have 4 components, so that the Dirac matrices and thus the generators $M^{\mu\nu}$ in the spin-1/2-representation can act on it. (Note that it is an accident that the spinor has as many components as there are spacetime dimensions. A Dirac spinor has four components in any spacetime dimensionality.).
  We can also argue for the four components on physical grounds: an electron is a massive spin 1/2 particle, and hence comes in a left-handed and a right-handed version, which makes up two degrees of freedom.[13] However, we already know from our experience with the complex scalar field, that a QFT needs particles and antiparticles in order for causality to hold (remember that the commutator of $\phi(x)$ and $\phi(y)$ at spacelike distances vanishes, but its two parts, the propagator from x to y and the commutator from y to x, do not vanish individually, but cancel in the commutator. This has the interpretation of the effect of a particle traveling from x to y cancelling against the effect of an antiparticle travelling from y to x.). Therefore, we also need the positron, which also has a right-handed component. Thus, we need a total of four degrees of freedom.
- it cannot be a 4-vector, because it does *not* transform in the 4-vector representation. The index that it carries must be a *new kind* of index, namely one associated to the spinor representation. We will use Latin indices from the *beginning* of the alphabet $(a, b, ..)$.
  (Some books also use Greek indices from the beginning of the alphabet $(\alpha, \beta...)$ as well as Greek letters where one puts a dot on top. In this convention, α, β will be the indices of the left-handed spin-1/2-representation, and $\dot{\alpha}$, $\dot{\beta}$ will be the indices of the right-handed spin-1/2-representation. Instead, we just work with indices $(a, b, \dots) = (1, 2, 3, 4)$ to address the four components of the Dirac spinor.)

The P_L and P_R, when acting on a Dirac spinor with 4 components, isolate the left-handed and the

[13] If the electron was massless, we could make do with just the "left-handed" or "right-handed" version, because a massless particle travels at the speed of light and hence one cannot "overtake" it by a boost and therefore its "handedness" (or rather, its chirality), remain unchanged.

right-handed spinor. We thus introduce a Dirac spinor, i.e., an object with four components

$$\psi_D = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{375}$$

The *defining property of a spinor is how it transforms under a Lorentz transformation, see below.* In the above expression, just like when we were writing Dirac matrices, we suppressed the spinor indices (also called Dirac indices). In general, we will use Latin letters from the beginning of the alphabet $(a, b, \dots)$ to denote the spinor indices. Thus, we could write ψ_D^a to make more explicit that ψ_D is an object that has four components, labelled by the spinor index.
When we act with the two projectors $P_{R,L}$, we isolate the upper or lower part of the Dirac-spinor, such that you will often find a notation such as

$$\psi_D = \begin{pmatrix} \xi \\ \chi \end{pmatrix} \tag{376}$$

in QFT lecture notes or books. Each of the two-spinors χ and ξ has two components, because it contains a particle and an antiparticle. Unlike for the scalar field, where we had a choice between a one-component version (where the particle is its own antiparticle and the field carries no charge) and a two-component version (more conveniently written as a complex field), for spinors we are naturally led to a version that contains antiparticles that we distinguish from particles. There is also a Majorana spinor, for which the particle is its own antiparticle. We will not introduce it here. Spinors are in general complex (so that they seem to have 8 degrees of freedom – double the number that is needed to describe a spin 1/2 particle and its spin 1/2 antiparticle. We will see below how this seeming problem is resolved).

Now we can write how a Lorentz transformation acts on a spinor:
Just like for a scalar field, in the active view of a transformation, we evaluate the field on an argument which is the *inverse* transformation, i.e., for a Lorentz transformation Λ, which is defined by its action on spacetime points, $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, we had $\phi(x) \to \phi(\Lambda^{-1}x)$.
For the scalar field, the factor $e^{i\omega^{\mu\nu}M_{\mu\nu}}$ was trivial, because the $M_{\mu\nu}$ are trivial in the scalar representation. This is no longer the case for spinors, just as it was not for vectors, where we had

$$A^\mu \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{377}$$

For spinors, we have

$$\psi_D \to e^{i\omega^{\mu\nu}M_{\mu\nu}}\psi(\Lambda^{-1}x), \tag{378}$$

where the $M_{\mu\nu}$ are $4x4$ matrices, defined through the Dirac matrices, that have the corresponding indices, i.e.,

$$e^{i\omega^{\mu\nu}M_{\mu\nu}}\psi(\Lambda^{-1}x) \to (\mathbb{1} + i\omega^{\mu\nu}M_{\mu\nu})\psi(\Lambda^{-1}x) + \mathcal{O}(\omega^2). \tag{379}$$

To get familiar with this, you might consider writing out a concrete example, e.g., a boost along the z-direction.
Let us emphasize again that the *definition of a spinor* is a four-component object that transforms

under a Lorentz transformation as in Eq. (379) with $M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ and $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}$. In other words, the definition of a spinor follows from the Clifford algebra.
Let us look at an example, namely an infinitesimal Lorentz transformation along the x-axis with infinitesimal boost parameter θ (which is the rapidity, with $\theta = v/c$ for small θ), just like in Eq. (42), which we repeat here for convenience:

$$\Lambda^\mu{}_\nu = \mathbb{1} + \begin{pmatrix} 0 & \theta & 0 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{380}$$

Given a four-vector-field $V^\mu(x)$ with components

$$V^\mu(x) = \begin{pmatrix} V_0(x) \\ V_1(x) \\ V_2(x) \\ V_3(x) \end{pmatrix}, \tag{381}$$

the transformed four-vector-field is

$$V^{\mu'}(x) = \Lambda^\mu{}_\nu V^\mu(\Lambda^{-1}x) = \begin{pmatrix} V_0(\Lambda^{-1}x) + \theta V_1(\Lambda^{-1}x) \\ V_1(\Lambda^{-1}x) + \theta V_0(\Lambda^{-1}x) \\ V_2(\Lambda^{-1}x) \\ V_3(\Lambda^{-1}x) \end{pmatrix} = \begin{pmatrix} V_0(\Lambda^{-1}x) + \frac{v}{c} V_1(\Lambda^{-1}x) \\ V_1(\Lambda^{-1}x) + \frac{v}{c} V_0(\Lambda^{-1}x) \\ V_2(\Lambda^{-1}x) \\ V_3(\Lambda^{-1}x) \end{pmatrix}, \tag{382}$$

which is the infinitesimal form of a boost, acting on a four-vector function (hence the argument of the function also transforms, and we work in the active view of a transformation; hence the transformation of the argument is with the inverse boost).
Now we consider a spinor field, which is also a four-component object,

$$\psi_D(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}, \tag{383}$$

but it does not transform according to Eq. (382). Instead, we need to calculate the form of M_{01} for the spinor-representation, and we must choose $\omega^{01} = -\omega^{10} = \theta$ (and all other components vanishing) and thus we need

$$M_{01} = \frac{i}{4}[\gamma_0, \gamma_1] = \frac{i}{4}\left(\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\right) = \frac{i}{2}\begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{384}$$

Then we can write

$$\psi'_D = \left(\mathbb{1} - \frac{\theta}{2}\begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\right)\begin{pmatrix} \psi_1(\Lambda^{-1}x) \\ \psi_2(\Lambda^{-1}x) \\ \psi_3(\Lambda^{-1}x) \\ \psi_4(\Lambda^{-1}x) \end{pmatrix} = \begin{pmatrix} \psi_1(\Lambda^{-1}x) + \frac{\theta}{2}\psi_2(\Lambda^{-1}x) \\ +\frac{\theta}{2}\psi_1(\Lambda^{-1}x) + \psi_2(\Lambda^{-1}x) \\ \psi_3(\Lambda^{-1}x) - \frac{\theta}{2}\psi_4(\Lambda^{-1}x) \\ -\frac{\theta}{2}\psi_3(\Lambda^{-1}x) + \psi_4(\Lambda^{-1}x) \end{pmatrix}. \tag{385}$$

This is clearly a different transformation rule than for a four-vector function. In particular, we see that the last two components are also affected by the Lorentz boost, unlike in the case of the four-vector function.
We do see, however, that the first two and the last two components of the Dirac spinor only mix among themselves. This is also true for all other Lorentz boost and for rotations. This is because a Dirac spinor transforms in a *reducible* representation of the Lorentz group and the last two and the first two components (that we can project onto with the two projection operators $P_{L/R}$) transform separately.

7.2 Dirac action and Dirac equation

To construct an action for spinor fields, we need real, Lorentz-invariant building blocks. A spinor is in general complex, because we in particular want to describe electromagnetically charged spin-1/2-particles like the electron and thus we need a field that transforms non-trivially under a U(1) transformation.[14]
Thus, we definitely will need a conjugate spinor $\psi^\dagger = (\psi^*)^\top$ to form a scalar together with ψ (similarly to a vector and its transposed vector forming a scalar product). Then, $\psi^\dagger\psi$ would be our first building block. To check whether it is, we need the behavior of $\psi^\dagger$ under a Lorentz transformation:

$$\psi(x) \to e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}\psi(\Lambda^{-1}x) \tag{386}$$

$$\implies (\psi^*)^\top(x) \to (\psi^*)^\top(\Lambda^{-1}x)e^{\left(\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right)^\dagger} = \psi^\dagger(\Lambda^{-1}x)e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu\dagger}}. \tag{387}$$

The $\top$ operation in $(\psi^*)^\top(x)$ is with respect to the *spinor* indices. We note that $\omega_{\mu\nu}$ is just a real number for any choice of μ, ν, therefore $(\frac{i}{2}\omega_{\mu\nu})^* = -\frac{i}{2}\omega_{\mu\nu}$. Furthermore, $M^{\mu\nu}$ is the matrix that carries the *spinor* indices and on which the †-operation acts nontrivially.
To understand how $\psi^\dagger\psi$ transforms, we therefore need to know $(M^{\mu\nu})^\dagger$:

$$M^{\mu\nu\dagger} = \left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger = -\frac{i}{4}\Big[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}\Big] = \frac{i}{4}\Big[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\Big]. \tag{388}$$

We used that $(AB)^\dagger = B^\dagger A^\dagger \implies [A, B]^\dagger = \big[B^\dagger, A^\dagger\big]$. From $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we know that $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$ (no summation over i implied!) and thus not all γ^μ can be hermitian at the same time, because $(\gamma^0)^2 = 1$ implies that γ^0 has real eigenvalues, but $(\gamma^i)^2 = -1$ implies that γ^i has imaginary eigenvalues.
Thus, $(M^{\mu\nu})^\dagger \neq M^{\mu\nu}$. Therefore

$$\psi^\dagger\psi \to \psi^\dagger e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu\dagger}} e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\psi \neq \psi^\dagger\psi, \tag{389}$$

because the factors inbetween do not cancel. Thus, we need something inbetween $\psi^\dagger$ and ψ. We note that $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$ (which can be checked by plugging in our representation for the γ's). Thus,

$$\gamma^0 M^{\mu\nu\dagger}\gamma^0 = \gamma^0\left(\frac{i}{4}[\gamma^\mu, \gamma^\nu]\right)^\dagger\gamma^0 \tag{390}$$

[14] Spinors that satisfy a reality condition (and thus do not carry any U(1) charge) are called Majorana spinors. We will not describe them here.

$$= \gamma^0 \left(-\frac{i}{4}\right) \left[\gamma^{\nu\dagger}, \gamma^{\mu\dagger}\right] \gamma^0 \tag{391}$$

$$= +\frac{i}{4} \gamma^0 \left[\gamma^{\mu\dagger}, \gamma^{\nu\dagger}\right] \gamma^0 \tag{392}$$

$$= \frac{i}{4} \left(\gamma^0 \gamma^{\mu\dagger} \gamma^{\nu\dagger} \gamma^0 - \gamma 0 \gamma^{\nu\dagger} \gamma^{\mu\dagger} \gamma^0\right) \tag{393}$$

$$= \frac{i}{4} \left(\gamma^0 \gamma^{\mu\dagger} \underbrace{\gamma^0 \gamma^0}_{=1} \gamma^{\nu\dagger} \gamma^0 - \gamma^0 \gamma^{\nu\dagger} \underbrace{\gamma^0 \gamma^0}_{=1} \gamma^{\mu\dagger} \gamma^0\right) \tag{394}$$

$$= \frac{i}{4} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \tag{395}$$

$$= M^{\mu\nu}. \tag{396}$$

$$\implies \boxed{\mathrm{M}^{\mu\nu\dagger} \gamma^0 = \gamma^0 M^{\mu\nu},} \tag{397}$$

using that $(\gamma^0)^2 = 1$. Based on this, we define

$$\overline{\psi} = \psi^\dagger \gamma^0. \tag{398}$$

Now we can show that $\overline{\psi}\psi$ transforms like a Lorentz scalar.

Mini-Exercise 21. Show that $\overline{\psi}\psi$ transforms like a Lorentz scalar.

Solution.

$$\overline{\psi} \to \psi^\dagger(\Lambda^{-1}x) e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu\dagger}} \gamma^0.$$

We expand the exponential and use $M^{\mu\nu\dagger}\gamma^0 = \gamma^0 M^{\mu\nu}$ in each term, then re-exponentiate again.

$$\begin{aligned}
\psi^\dagger(\Lambda^{-1}x) e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu\dagger}} \gamma^0 &= \psi^\dagger(\Lambda^{-1}x) \gamma^0 e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}} \\
&= \overline{\psi}(\Lambda^{-1}x) e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}} \\
\implies \overline{\psi}\psi \to \overline{\psi}(\Lambda^{-1}x) &\underbrace{e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}} e^{\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}}}_{=\mathbb{1}} \psi(\Lambda^{-1}x) = \overline{\psi}\psi(\Lambda^{-1}x).
\end{aligned}$$

Based on this, we might consider

$$\overline{\psi}\partial^2\psi, \tag{399}$$

or, equivalently (up to partial integration),

$$\partial_\mu \overline{\psi} \partial^\mu \psi, \tag{400}$$

as the kinetic term for spinors. There is one challenge with this, namely the count of degrees of freedom: To describe a charged spin-1/2-particle, we need four degrees of freedom (left-and right-handed for particle and antiparticle). However, a four-component complex field has eight degrees of freedom, if we describe it with second-order equations of motion, because then we expect to have two sets of creation and annihilation operators for each of the four components. Therefore, we start wondering whether we can have equations of motion that are just linear in derivatives,

because this will reduce the number of degrees of freedom by a factor of two.[15]
There is, in fact a possibility, which is *linear* in derivatives. This uses that

$$\overline{\psi}\gamma^\mu\psi \qquad \text{transforms as a Lorentz vector} \tag{401}$$

and hence $\overline{\psi}\partial_\mu\gamma^\mu\psi$ transforms like a scalar. To see that $\overline{\psi}\gamma^\mu\psi$ transforms as a vector, we need

$$[M_{\mu\nu},\gamma_\rho] = -\big(M^{(\text{fund})}_{\mu\nu}\big)_\rho{}^\sigma\gamma_\sigma, \tag{402}$$

where $M^{(\text{fund})}_{\mu\nu}$ are the generators in the fundamental representation (cf. Eq. (44)) which will be shown in the exercises.
Based on this, it follows that

$$e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}}\gamma_\rho\, e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} = (\Lambda^{-1})_\rho{}^\sigma\gamma_\sigma. \tag{403}$$

Given that $\partial^{\mu'} = \Lambda^\mu{}_\nu\partial^\nu$, it follows that $\overline{\psi}\partial^\mu\gamma_\mu\psi$ transforms as a Lorentz scalar.
If we also want to add a mass-term, we use that $\overline{\psi}\psi$ is a scalar, and thus we can write the Dirac Lagrangian:

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{404}$$

We introduce the shorthand notation $\not{v}$ "v-slash" for $v^\mu\gamma_\mu$, so that $\gamma^\mu\partial_\mu = \not{\partial}$ ("d-slash").
The Dirac action is accordingly given by

$$S_D = \int d^4x\,\overline{\psi}(i\not{\partial} - m)\psi. \tag{405}$$

The i is needed for S to be real. To see that it is needed, take the complex conjugate of the action. In doing so, i becomes $-i$ and the extra minus is needed to compensate another minus that arises from a partial integration, which must be done to bring S^* back into the form of S.
The Dirac equation is the equation of motion that follows. Because ψ is complex, we treat ψ and $\psi^\dagger$ as independent and

$$0 = \frac{\delta S_D}{\delta\psi^\dagger(y)} = \int d^4x\,\delta^4(x-y)\gamma^0(i\not{\partial} - m)\psi \implies \boxed{(i\not{\partial} - m)\psi = 0}. \tag{406}$$

This is the Dirac equation. It is the "square-root" of the Klein-Gordon equation in the sense that

$$(i\gamma^\mu\partial_\mu + m)\underbrace{(i\gamma^\nu\partial_\nu - m)\psi = 0}_{\text{Dirac equation}} \tag{407}$$

$$\implies \big(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - \cancel{i\gamma^\mu\partial_\mu m} + \cancel{mi\gamma^\nu\partial_\nu} - m^2\big)\psi = 0 \tag{408}$$

Because $\partial_\mu\partial_\nu$ is symmetric under exchange of the indices, we have

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\gamma^\mu\gamma^\nu(\partial_\mu\partial_\nu + \partial_\nu\partial_\mu) \tag{409}$$

[15] When we considered the Ostrogradsky theorem, we already saw the opposite result (in the exercises), namely that for equations of motion that are of third order in time derivatives, there is one additional degree of freedom per real field.

$$= \frac{1}{2}\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \underbrace{\gamma^\mu\gamma^\nu\partial_\nu\partial_\mu}_{\text{relabeling } \mu\leftrightarrow\nu}\right) \tag{410}$$

$$= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu \tag{411}$$

$$= \frac{1}{2}2\eta^{\mu\nu}\partial_\mu\partial_\nu \tag{412}$$

$$= \partial^2. \tag{413}$$

Therefore,

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\nu\partial_\nu - m)\psi = (\partial^2 - m^2)\psi = 0. \tag{414}$$

Thus, a solution of the Dirac equation also constitutes a solution of the Klein-Gordon equation.

7.3 Solutions of the Dirac equation

When quantizing and defining creation and annihilation operators, we always expand the field in a complete set of classical solutions (in Fourier space). We did this for the scalar field, where $\phi_k \sim e^{\pm ikx}$ with $k^2 = m^2$ constitute the solutions to the Klein-Gordon equation; and we did this for the gauge field, where $A_{\mu k} \sim \varepsilon_\mu^{(1,2)} e^{\pm ikx}$ with $k^2 = 0$ constitute the solutions to the Maxwell equations.
Similarly, we need a complete set of solutions to the Dirac equation to expand the field in terms of creation and annihilation operators and quantize. Because any solution to the Dirac equation also solves the Klein-Gordon equation, we know that the solutions contain plane waves $e^{\pm ikx}$. However, just like for the photon field, we need the polarization vectors to take care of the spacetime index μ of A_μ, we need to figure out how the spinor index is constrained by the requirement f solving the Dirac equation.

We make the ansatz

$$\psi(x) = u(p)e^{-ipx} \qquad \text{with } p^2 = m^2 \tag{415}$$

and choose $p^0 > 0$ (note that if $p^0 > 0$ or $p^0 < 0$ are both treated simultaneously, e^{-ipx} gets mapped to e^{ipx}). Below, we will consider the solutions e^{ipx} (with $p^0 > 0$) separately. $u(p)$ is a spinor, i.e., it has 4 components and a spinor index. Just like for the gauge field, the requirement of solving the Maxwell equations constrains the structure of polarization vectors (e.g., making them transverse to the direction of propagation), the requirement of solving the Dirac equation constrains $u(p)$. The Dirac equation reads:

$$0 = (i\partial\!\!\!/ - m)\psi(x) = (i(-ip\!\!\!/) - m)\, u(p)e^{-ipx} \tag{416}$$

$$\implies (p\!\!\!/ - m)u(p) = 0. \tag{417}$$

Re-instating Dirac indices (for which we use Latin letters), we have

$$(p_\mu\gamma^\mu_{ab} - m\mathbb{1}_{ab})u^b(p) = 0. \tag{418}$$

This makes it clear, that $(p\!\!\!/ - m)u(p) = 0$ imposes a non-trivial requirement that relates the 4 components of the spinor to each other. To figure out what this looks like, we use Lorentz

invariance to our advantage: We go to a frame in which $p = (m, \vec{0})$, where

$$(\not{p} - m)u(p) = 0 \implies m(\gamma^0 - \mathbb{1}_4)u(p) = 0. \tag{419}$$

$\mathbb{1}_4$ is a 4×4 matrix. In terms of 2×2 matrices $\mathbb{1}_{2x2}$, this reads

$$\begin{pmatrix} -\mathbb{1}_{2x2} & \mathbb{1}_{2x2} \\ \mathbb{1}_{2x2} & -\mathbb{1}_{2x2} \end{pmatrix} u(p) = 0. \tag{420}$$

The fact that the previous equation is built out of 2×2 matrices suggests that it makes sense to write

$$u(p) = \begin{pmatrix} \xi \\ \xi' \end{pmatrix}, \qquad \text{where } \xi \text{ and } \xi' \text{ are both 2-component spinors.} \tag{421}$$

(These are the Weyl spinors). Then $m(\gamma^0 - \mathbb{1})u(p) = 0$ implies a relation between ξ and ξ'.

Mini-Exercise 22. What is the relation between ξ and ξ'?

Solution.

$$\begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \xi \\ \xi' \end{pmatrix} = 0$$
$$\implies -\xi + \xi' = 0$$

In other words, to have a solution to the Dirac equation, the first two and last two components of the Dirac spinor have to be related to each other. We remember that these are the left-handed and the right-handed irreducible spin-1/2-representations of the Lorentz group, respectively. It makes sense that if we're putting them together to describe a spin-1/2-field with mass (such that, dependent on the Lorentz frame, the spin is either aligned with the direction of travel (right-handed) or anti-aligned (left-handed)), that we then cannot choose the components independently, but they have to be related to each other.

The 4 components are the particles (e.g., electron), in both left- and right-handed version, and its antiparticles (e.g., the positron), in both the left- and right-handed version. (For a spinor that is its own antiparticle, we can correspondingly use 2-component (Majorana) spinors.)

We proceed with the explicit construction of the solutions. Since the space of 2-spinors is a 2-d-space (because they have two independent components), we have

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{422}$$

such that

$$u_1(p) \sim \begin{pmatrix} \xi_1 \\ \xi_1 \end{pmatrix} \quad \text{and} \quad u_2(p) \sim \begin{pmatrix} \xi_2 \\ \xi_2 \end{pmatrix}. \tag{423}$$

We use the Latin letter s to denote the choice between ξ_1 and ξ_2, i.e., we write $u_s(p)$, $s = 1, 2$.

It is convenient to normalize

$$u_s(p) = \sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix} \tag{424}$$

in the frame in which $p = (m, \vec{0})$.
The solution in any other frame can be obtained by the corresponding Lorentz transformation acting on $u_s(p)$. Now we still have the second set of solutions, with e^{+ipx} and $p^0 > 0$. These are called negative-frequency solutions, whereas e^{-ipx} are the positive frequency solutions. The e^{+ipx} have *negative* energy if we plug them into the expression for the conserved energy from Noether's theorem.
We write these solutions as

$$\psi(x) = v(p)e^{ipx}, \qquad p^2 = m^2,\ p^0 > 0. \tag{425}$$

In this case

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \implies (p\!\!\!/ + m)v(p) = 0, \tag{426}$$

i.e., there is a difference in the relative sign between $p\!\!\!/$ and m compared to the u's.
In matrix notation

$$\begin{pmatrix}\mathbb{1} & \mathbb{1}\\ \mathbb{1} & \mathbb{1}\end{pmatrix} v(p) = 0. \tag{427}$$

Hence we can write

$$v_s(p) = \sqrt{m}\begin{pmatrix}\eta_s\\ -\eta_s\end{pmatrix}, \qquad s = 1, 2 \tag{428}$$

with

$$\eta_1 = \begin{pmatrix}1\\ 0\end{pmatrix} \quad \text{and} \quad \eta_2 = \begin{pmatrix}0\\ 1\end{pmatrix} \tag{429}$$

(in a frame where $p = (m, \vec{0})$; the solution in any other frame is obtained by the corresponding Lorentz transformation.)
Note that the different sets of solutions satisfy orthonormality relations (which we will check in the exercises):

$$\overline{u}_r(p)u_s(p) = 2m\delta_{rs} \tag{430}$$

$$\overline{v}_r(p)v_s(p) = -2m\delta_{rs} \tag{431}$$

$$\overline{u}_r(p)v_s(p) = 0 \tag{432}$$

$$\overline{v}_r(p)u_s(p) = 0. \tag{433}$$

The factor of $2m$ is due to our choice of normalization.
Note: We constructed the explicit expression for the spinors in one particular frame, and can also check the left-hand-side of the orthonormality relations in that frame. The right-hand-side is a Lorentz scalar and this the orthonormality relations hold in this form in all frames.

In addition to these orthonormality relations, there is also a form of "completeness relation", in

which $(\not{p}+m)$ and $(\not{p}-m)$ are rewritten in terms of u's and v's:

$$(\not{p}+m)_\alpha{}^\beta = \sum_{s=1}^{2} (u_s(p))_\alpha (\overline{u}_s(p))^\beta. \tag{434}$$

α and β are spinor indices. We interpret $(u_s(p))_\alpha$ as a column vector and $(\overline{u}_s(p))^\beta$ as a row vector, therefore their product is a *matrix*.
Similarly,

$$\sum_{s=1}^{2} v_s(p)\overline{v}_s(p) = \not{p} - m, \tag{435}$$

where we suppress the Dirac indices, but the relation is again between two objects that are both *matrices* in Dirac indices.

7.4 Quantization of spinors

Here, we will only perform the canonical quantization of spinors. The path-integral quantization will be done in QFT II and will require us to introduce a new type of numbers, namely *anticommuting* numbers (called Grassmann numbers).
For the canonical quantization, we first need the Hamiltonian. Starting from

$$\mathcal{L} = \overline{\psi}(i\not{\partial} - m)\psi, \tag{436}$$

it follows that

$$\Pi^a = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^a} = \frac{\partial}{\partial \dot{\psi}^a}\left(i\psi^\dagger \gamma^0 \gamma^0 \dot{\psi}\right) = i(\psi^\dagger(\gamma^0)^2)^a = i(\psi^\dagger)^a. \tag{437}$$

We note that the Lagrangian can be expressed in terms of ψ and Π without the need to introduce a canonically conjugate field for $\psi^\dagger$. This is different from the complex scalar field, where a canonically conjugate field was needed for ϕ and for ϕ^*, reflecting the fat that there are *two* sets of degrees of freedom (associated to particle and antiparticle). In the case of the spinor field, the fact that the Dirac equation is first order results in a reduction of degrees of freedom. This is exactly what we need - if we had the full eight degrees of freedom associated to a complex 4-component field, that would be too many degrees of freedom.

Thus, the Hamiltonian density is

$$\mathcal{H} = \Pi\dot{\psi} - \mathcal{L} = i\psi^\dagger\dot{\psi} - \psi^\dagger\gamma^0(i\not{\partial} - m)\psi \tag{438}$$

$$= -\psi^\dagger\gamma^0(i\gamma^i\partial_i - m)\psi \tag{439}$$

$$= i\Pi\,\gamma^0(i\gamma^i\partial_i - m)\psi. \tag{440}$$

Now, we expand the field in terms of creation and annihilation operators:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}} \sum_{s=1,2}\left(a^s_{\vec{p}}\,u_s(p)e^{-ipx} + b^{s\dagger}_{\vec{p}}\,v_s(p)e^{ipx}\right) \tag{441}$$

and now have the choice between imposing commutation or anticommutation relations.
Let us first summarize what goes wrong if we use commutators:

1) Imposing

$$[\psi(\vec{x}), \Pi(\vec{y})] = \left[\psi(\vec{x}), i\psi^\dagger(\vec{y})\right] = i\delta^3(\vec{x}-\vec{y})\mathbb{1} \tag{442}$$

($\mathbb{1}$ in spinor indices) is not consistent with

$$\left[a^s_{\vec{p}}, a^{r\dagger}_{\vec{q}}\right] = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta^{rs}\cdot 2\omega_{\vec{p}} = \left[b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\right]. \tag{443}$$

If we express $\psi(\vec{x})$ and $\psi^\dagger(\vec{x})$ through a's and b's and calculate $\left[\psi(\vec{x}), i\psi^\dagger(\vec{y})\right]$ by using Eq. (443), we do *not* obtain Eq. (442). The completeness relation for the u's and v's is crucial here:

$$\left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] =$$

$$\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\sum_{r,s}\left(e^{i\vec{p}\cdot\vec{x}}e^{-i\vec{q}\cdot\vec{y}}u_s(p)u^\dagger_r(q)\left[a^s_{\vec{p}}, a^{r\dagger}_{\vec{q}}\right] + e^{-i\vec{p}\cdot\vec{x}}e^{i\vec{q}\cdot\vec{y}}v_s(p)v^\dagger_r(q)\left[b^{s\dagger}_{\vec{p}}, b^r_{\vec{q}}\right]\right) \tag{444}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{(2\omega_{\vec{p}})^2}\sum_{s,r}\left(u_s(p)u^\dagger_r(p)\delta_{rs}e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - v_s(p)v^\dagger_r(p)\delta_{rs}e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right)\cdot 2\omega_{\vec{p}}. \tag{445}$$

Now we use

$$\sum_s u_s(p)u^\dagger_s(p) = (\not{p}+m)\gamma_0 \qquad \text{and} \tag{446}$$

$$\sum_s v_s(p)v^\dagger_s(p) = (\not{p}-m)\gamma_0, \tag{447}$$

which follow from the completeness relations. Therefore,

$$\left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})}(\not{p}+m) - \underbrace{e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}(\not{p}-m)}_{\vec{p}\to-\vec{p}}\right)\gamma^0 \tag{448}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2\omega_{\vec{p}}}\left(\cancel{p_0\gamma^0} + p_i\gamma^i + m - \left(\cancel{p_0\gamma^0} - p_i\gamma^i - m\right)\right)\gamma^0. \tag{449}$$

We note that $p_0\gamma^0$ is unchanged by the sign flip, whereas $p_i\gamma^i$ is changed by the sign flip. Finally, we arrive at

$$\left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] = \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{i\vec{p}\cdot(\vec{x}-\vec{y})}}{2\omega_{\vec{p}}}\underbrace{\left(2p_i\gamma^i + 2m\right)}_{(\star)}\gamma^0 \neq i\delta^3(\vec{x}-\vec{y})\mathbb{1}, \tag{450}$$

which we postulated for this commutator. We would have needed the p^0-term to *cancel* $\omega_{\vec{p}}$, but instead it dropped out. The terms marked with $(\star)$ prohibit us from forming a δ-function.

The problem lies with the b-terms and can be solved if we assume that

$$\left[b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\right] = -(2\pi)^3\delta^3(\vec{p}-\vec{q})\delta^{rs}\cdot 2\omega_{\vec{p}}, \tag{451}$$

i.e. we introduce an additional sign, which effectively amounts to exchanging the roles of b and $b^\dagger$.

2) Going into the calculation of $H = \int d^3x\, \mathcal{H}$ with the b-commutator with negative sign, we find

$$H = \int \frac{d^3p}{(2\pi)^3}\, \omega_{\vec{p}} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right). \tag{452}$$

The energy is unbounded form below, which means that the ground state is unstable. This is a serious problem and means that we have to find a different way of quantizing.

We next try anticommutators, i.e., we demand

$$\{\psi(\vec{x}), \Pi(\vec{y})\} = \psi(\vec{x})\Pi(\vec{y}) + \Pi(\vec{y})\psi(\vec{x}) = i\delta^3(\vec{x} - \vec{y})\mathbb{1}. \tag{453}$$

Then, the creation/annihilation operators satisfy

$$\left\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}} \right\} = \left\{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}} \right\} = (2\pi)^3 2\omega_{\vec{p}} \delta^3(\vec{p} - \vec{q}) \delta^{rs} \tag{454}$$

and the other anticommutators are zero. You can check that Eq. (453) and Eq. (454) are consistent with each other.
In addition, we also find a Hamiltonian that is bounded from below:

$$H = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2\omega_{\vec{p}}} \omega_{\vec{p}} \sum_s \left(a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \right), \tag{455}$$

where we have dropped an infinite constant.
But what is the physical meaning of what we have done? We can see that, when we built the Fock space, starting from the ground state $|0\rangle$ for which $a^s_{\vec{p}}|0\rangle = 0 = b^s_{\vec{p}}|0\rangle$, an interesting result holds:

Mini-Exercise 23. 1) What is $\left(a^{s\dagger}_{\vec{p}}\right)^2 |0\rangle$?

2) What is $a^{s\dagger}_{\vec{p}_1} a^{s\dagger}_{\vec{p}_2} |0\rangle + a^{s\dagger}_{\vec{p}_2} a^{s\dagger}_{\vec{p}_1} |0\rangle$?

3) What do your results mean at the physical level?

Solution.

1)

$$\left\{ a^{s\dagger}_{\vec{p}}, a^{r\dagger}_{\vec{q}} \right\} = 0 \implies \left(a^{s\dagger}_{\vec{p}}\right)^2 = \frac{1}{2}\left\{ a^{s\dagger}_{\vec{p}}, a^{s\dagger}_{\vec{p}} \right\} = 0 \implies \left(a^{s\dagger}_{\vec{p}}\right)^2 |0\rangle = 0.$$

→ This is the Pauli-exclusion principle. We cannot have two particles with the same spin ($s = 1$ or $s = 2$) and the same momentum.

2)

$$\left\{ a^{s\dagger}_{\vec{p}_1}, a^{s\dagger}_{\vec{p}_2} \right\} = 0 \implies |\vec{p}_1, s;\, \vec{p}_2, s\rangle = a^{s\dagger}_{\vec{p}_1} a^{s\dagger}_{\vec{p}_2} |0\rangle = -\,|\vec{p}_2, s;\, \vec{p}_1, s\rangle\,,$$

i.e., the states are *antisymmetric* under the exchange of two particles.

3) We are dealing with *fermions*, i.e., particles which obey the Pauli exclusion principle and

more general antisymmetry of the state under the exchange of any two particles (from which the Pauli principle follows).

Thus, we have that:

- spin-1/2-fields must be quantized with anti-commutators and cannot be quantized with commutators. At the physical level, this mathematical property implies that spin-1/2-fields give rise to fermions, i.e., particles have wavefunctions that are antisymmetric under exchanges of two particles and obey the Pauli principle.
- We already say that the correct way to quantize scalar fields and vectors (which have integer spin- 0 and 1, respectively) is with commutators.

We will not prove it here, but there is the *spin-statistics theorem* that states:
(Half-)integer spin fields must be quantized with (anti-)commutators. As a consequence, particles corresponding to half-integer spin fields are fermions, and particles corresponding to integer-spin fields are bosons.

7.5 $U(1)$ symmetry of the Dirac Lagrangian

When discussing the degrees of freedom of the Dirac spinor ψ_D, which satisfy the Dirac equations, we already used that it describes a spin-1/2-particle together with its antiparticle. But what is actually the associated charge and the corresponding symmetry?
It turns out that $\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under a global $U(1)$ symmetry, under which

$$\psi \to e^{-i\alpha}\psi = \psi', \quad \overline{\psi} \to e^{+i\alpha}\overline{\psi} = \overline{\psi}', \tag{456}$$

and, infinitesimally,

$$\delta\psi = \frac{1}{\alpha}(\psi' - \psi) = -i\psi \tag{457}$$

such that

$$\mathcal{L}' = \overline{\psi}'(i\partial\!\!\!/ - m)\psi' = \mathcal{L}. \tag{458}$$

Noether's theorem states that

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \underbrace{\delta\psi}_{=-i\psi} - (\underbrace{\mathcal{L}' - \mathcal{L}}_{=0}) \tag{459}$$

$$= \overline{\psi}i\gamma^\mu(-i\psi) = \overline{\psi}\gamma^\mu\psi \tag{460}$$

is conserved. If we decide to make the symmetry *local*, $\psi \to e^{i\alpha(x)}\psi$, we will find that the local transformation is *not* a symmetry, unless we couple the fermions to a gauge field. When we link the U(1) symmetry of the fermions to the gauge symmetry of the gauge field, we are free to introduce a dimensionless coupling constant e in the transformation of the fermion, so that

$$\psi(x) \to e^{i\,e\,\alpha(x)}\psi(x). \tag{461}$$

Then,

$$\mathcal{L}_D = \overline{\psi}i(\partial\!\!\!/ + ie\gamma^\mu A_\mu - m)\psi = \overline{\psi}i(\partial\!\!\!/ + ieA\!\!\!/ - m)\psi \tag{462}$$

is invariant under a local $U(1)$. We can see that e governs the strength of the interaction between ψ and A_μ, and that, as $e \to 0$ the interaction is switched off (as is the transformation of ψ under the U(1) in this limit).
The complete Lagrangian for Quantum Electrodynamics reads

$$\mathcal{L}_{\text{QED}} = \overline{\psi} i(\partial\!\!\!/ + ieA\!\!\!/ - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{463}$$

We already know how to quantize the two individual free parts, namely the Dirac Lagrangian and the Lagrangian for electrodynamics. However, for the interaction term, $e\,\bar{\psi} A\!\!\!/ \psi$, we need to go beyond the quantization of the free theory. One possibility would be to do this in the path-integral formalism, but we have not discussed fermions in the path-integral formalism yet.
Instead, we will now develop *scattering* theory to describe one effect of interactions, namely that they give rise to scattering of particles off each other.

8 Scattering Theory, perturbation theory and Feynman rules

One of the key goals of QFT is to predict the results of scattering experiments, e.g., at the Large Hadron Collider (LHC). For non-trivial scattering, interactions must be present. We will develop the corresponding formalism now. Whenever we need a specific example, we will consider $\lambda\phi^4$-interactions, i.e., we will use the scalar Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{464}$$

as our "workhorse".

We add a ϕ^4 term, because it is the next term in a Taylor expansion of the potential $V(\phi)$, if we restrict to $\mathbb{Z}_2$-symmetry $\phi \to -\phi$ in order to prevent uneven terms that make the potential unbounded from below, such as ϕ^3 or ϕ^5.
At this point, our choice of interaction may seem a bit ad-hoc, because there seem to be so many other choices (e.g., $\partial_\mu\phi\partial^\mu\phi^2$, or ϕ^6, or $(\partial_\mu\phi\partial^\mu\phi)^2$). In QFT II, when we learn about the Renormalization Group and about effective field theories, we will see that $\lambda\phi^4$ generically is the *dominant* term at energies far below the Planck scale, even if other terms are present at higher energies.
In short, this can be motivated from *dimensional analysis*: In $\hbar = 1 = c$, we can assign dimensions of *mass* (or energy – these have equal dimension in these units) to everything. For instance, the action must be dimensionless (i.e., have no units). We write angular brackets around a quantity to indicate the dimension of this quantity, e.g., $[S] = 0$. Now let us consider the building blocks of the action: In units $\hbar = 1 = c$, $[d^4x] = \text{mass}^{-4}$. Therefore, the Lagrangian must have dimension 4. Let us first consider the kinetic term. Each derivative has dimension 1, and from this, we can infer the dimension of the field:

$$[\phi] = \frac{4-2}{2} = 1, \tag{465}$$

where the 4 is the dimension of the Lagrangian, and we are subtracting a 2, because each of the derivatives contributes with dimension 1. Finally, we divide by 2, because there are two fields.

Mini-Exercise 24. What is the dimension of the mass, m^2? What is the dimension of λ? And what is the dimension of λ_n in a higher-order interaction term $\lambda_n\phi^n$, with $n > 4$?

Solution. $[m^2] = 4 - 2 = 2$, which is consistent with the fact that this quantity is supposed to be the square of a mass.
$[\lambda] = 4 - 4 = 0$.
$[\lambda_n] = 4 - n$, which is negative for $n > 4$.

Let us now imagine that we are computing the contribution of λ and λ_n to a scattering process, in which the center-of-mass-energy is E. We express both their contributions through *dimensionless* parameters, in order to be able to compare them to each other. λ is already dimensionless, so that is simple, but λ_n is not. How can we make it dimensionless? We have the center-of-mass energy E at our disposal, and can write $\lambda_n \cdot E^{n-4}$ to obtain a dimensionless quantity. We see that if E is large, this contribution becomes large, but, if E is small, this contribution is small. This, in

essence, is, why we work with the *couplings of highest mass dimension*. At low enough energies, these are the dominant terms in the Lagrangian.
These arguments will be formalized and made more precise when we study the Renormalization Group in QFT II. There, we will see that the Standard Model Lagrangian is not some arbitrary choice, but contains *all* highest-dimension interactions between the existing fields. In other words, it is the *generic* approximation at low energies, irrespective of what other interactions are important at higher energies.

8.1 Interaction picture

We will work in the interaction picture for some parts of what we will do below. In analogy to the Heisenberg picture, it can be derived from the Schrödinger picture.

Let's first remind ourselves how that worked in Quantum Mechanics:
The Hamiltonian is split into two parts, a free part H_0 and an interaction part H_{int}, so $H = H_0 + H_{\text{int}}$. Then

$$\mathcal{O}^I(t) = e^{iH_0 t}\mathcal{O}e^{-iH_0 t} \tag{466}$$

is the time evolution of an operator $\mathcal{O}$ in the interaction picture, given the time-independent operator $\mathcal{O}$ in the Schrödinger picture. In other words, operators evolve with the free Hamiltonian. For the state, we start with the state at $t = 0$ in the Schrödinger picture and write

$$\left|\psi^I(t)\right\rangle = e^{iH_0 t}e^{-iHt}\left|\psi\right\rangle, \tag{467}$$

with $|\psi\rangle = |\psi(t=0)\rangle$ in the Schrödinger picture.

In this way, expectation values and other measurable quantities have the same time-evolution as in the Schrödinger picture (as they have to!), namely

$$\left\langle\psi^I\middle|\mathcal{O}^I\middle|\psi^I\right\rangle = \langle\psi|e^{iHt}\underbrace{e^{-iH_0 t}e^{iH_0 t}}_{=1}\mathcal{O}\underbrace{e^{-iH_0 t}e^{iH_0 t}}_{=1}e^{-iHt}|\psi\rangle. \tag{468}$$

In our case of ϕ^4-theory, we define

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}, \tag{469}$$

i.e., $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$ and $\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}\phi^4$ and accordingly

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \tag{470}$$

such that $\mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\phi^4$ (which is identical to $\mathcal{L}_{\text{int}}$ up to a sign).

We are now interested in the time-evolution from any time t to another time t', in the presence of this interaction. To this end, we write the time evolution of the state from 0 to t' and from t to 0 separately:

$$\left|\psi^I(t')\right\rangle = e^{iH_0 t'}e^{-iHt'}\left|\psi\right\rangle \tag{471}$$

$$\left|\psi^I(0)\right\rangle = e^{iHt}e^{-iH_0 t}\left|\psi^I(t)\right\rangle. \tag{472}$$

In combination:

$$\left|\psi^I(t')\right\rangle = \underbrace{e^{iH_0t'}e^{-iHt'}e^{iHt}e^{-iH_0t}}_{U(t',t)}\left|\psi^I(t)\right\rangle. \tag{473}$$

$U(t',t) = e^{iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}$ is the unitary time-evolution operator.
Of course we would like to summarize the factors in the several different exponentials into one. To do so, we can proceed very similarly to one of the steps we used in the derivation of the path integral, and split the interval from t to t' into n steps of duration $\Delta = \frac{t'-t}{n}$. Then

$$U(t',t) = U(t',t'-\Delta)\cdot U(t'-\Delta,t'-2\Delta)\ldots U(t+\Delta,t). \tag{474}$$

In each individual step, we can use the Baker-Campbell-Hausdorff (BCH) formula

$$e^{iH_0\Delta}e^{-iH\Delta} = e^{i(H_0-H)\Delta+\mathcal{O}(\Delta^2)} = e^{-iH_{\text{int}}\Delta+\mathcal{O}(\Delta^2)} \tag{475}$$

and neglect higher-order corrections in Δ in the exponential, because at each order of the Taylor-expansion of the exponential, these terms will contribute at higher order in Δ.
This allows us to rewrite

$$\begin{aligned} U(t+\Delta,t) &= e^{iH_0(t+\Delta)}e^{-iH\Delta}e^{-iH_0t} && (476)\\ &= e^{iH_0t}e^{iH_0\Delta}e^{-iH\Delta}e^{-iH_0t} && (477)\\ &\underset{\text{BCH}}{\approx} e^{iH_0t}e^{-iH_{\text{int}}\Delta}e^{-iH_0t} = e^{-iH^I_{\text{int}}(t)\Delta}. && (478)\end{aligned}$$

In the last step, we also introduced the interaction Hamiltonian in the interaction picture,

$$H^I_{\text{int}}(t) = e^{iH_0t}H_{\text{int}}e^{-iH_0t}. \tag{479}$$

In total we get (note that $H^I_{\text{int}}(t)$ always appears with an argument; the parentheses indicate functional dependence and do not denote multiplication):

$$\begin{aligned} U(t',t) &\approx e^{-iH^I_{\text{int}}(t'-\Delta)\cdot\Delta}e^{-iH^I_{\text{int}}(t'-2\Delta)\cdot\Delta}\ldots e^{-iH^I_{\text{int}}(t)\cdot\Delta} && (480)\\ &\approx Te^{-iH^I_{\text{int}}(t'-\Delta)\cdot\Delta}\ldots e^{-iH^I_{\text{int}}(t)\cdot\Delta} && (481)\\ &\approx Te^{-iH^I_{\text{int}}(t'-\Delta)\cdot\Delta-\ldots-iH^I_{\text{int}}(t)\cdot\Delta} && (482)\\ &= Te^{-i\int_t^{t'}dt''H^I_{\text{int}}(t'')}. && (483)\end{aligned}$$

The final result is called *Dyson's formula:*

$$U(t',t) = Te^{-i\int_t^{t'}dt''H^I_{\text{int}}(t'')}. \tag{484}$$

We introduced a time-ordering operator here, that we already used when discussing time-ordered correlation functions. We recall the definition here:

$$T\mathcal{O}(t_1)\mathcal{O}(t_2) = \begin{cases}\mathcal{O}(t_1)\mathcal{O}(t_2), & t_1 \geqslant t_2\\ \mathcal{O}(t_2)\mathcal{O}(t_1), & t_2 > t_1.\end{cases} \tag{485}$$

In our case

$$H^I_{\rm int} = e^{iH_0 t} H_{\rm int} e^{-iH_0 t} = e^{iH_0 t} \int d^3x \underbrace{\frac{\lambda}{4!}(\phi(\vec{x}))^4}_{\mathcal{H}_{\rm int}} e^{-iH_0 t} = \int d^3x \, \frac{\lambda}{4!}\big(\phi^I(\vec{x})\big)^4, \tag{486}$$

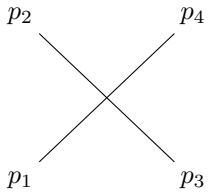
where $\phi^I(\vec{x})$ denotes the interaction-picture field. In the last step we introduced the interaction-picture field. Note that because interaction-picture operators evolve just with the free Hamiltonian, the interaction-picture field is the same as the Heisenberg-picture field. This will allow us to use some of our results from our study of the free theory.
In Dyson's formula, we can expand the final exponential and write

$$U(t,t') = 1 - i\int_t^{t'} dt'' H^I_{\rm int}(t'') + (-i)^2 \int_t^{t'} dt'' \int_t^{t''} dt''' \, H^I_{\rm int}(t'') \, H^I_{\rm int}(t''') + ... \tag{487}$$

This is ultimately a formal power series in λ. The first non-trivial term, $\sim \lambda$, contains four powers of ϕ^I. This operator contains a and $a^\dagger$ and can thus create or destroy one particle. Therefore, $\lambda(\phi^I)^4$ gives rise to the scattering of three particles into one (or one into three), or, most commonly considered (see below) two into two.
Now, we are ready to discuss *scattering*.

8.2 Scattering matrix or S-matrix: a first glimpse

We will now develop the theoretical description of scattering events, which will lead us to the Feynman diagrams. Feynman diagrams are actually diagrammatic ways of encoding a precise calculation, but they also have the advantage that they are quite intuitive. For instance,



denotes a Feynman diagram in which two particles (with momentum p_1 and p_2, respectively), scatter into two other particles with momenta p_3 and p_4. Time increases from left to right in Feynman diagrams. However, the Feynman diagram is not a *sketch of the process*, instead, it is a graphical way of depicting the calculation of the corresponding probability amplitude, and thus each element of such a diagram has a rule attached to it that allows us to go back and forth between an equation and a diagram. We will derive these rules over the coming lectures.

Before doing so, let us describe scattering more generally. We start with 2-2-scattering. This is motivated by the interaction term that we are considering, $\lambda \int d^4x \, \phi^4$, which includes terms of the type $a^\dagger_{\vec{p}_{1'}}$, $a^\dagger_{\vec{p}_{2'}}$, $a_{\vec{p}_1}$, $a_{\vec{p}_2}$, which annihilate a particles with $\vec{p}_1$ and one with $\vec{p}_2$, and in their place create two new particles, one with $\vec{p}_{1'}$ and one with $\vec{p}_{2'}$. (We already conjecture that at higher order in λ, we can describe 2-n-scattering, with $n > 2$. We also saw this in our study of the effective potential, where a ϕ^6 term was generated at $\mathcal{O}(\lambda^3)$.

In order to be able to describe a *scattering* event, i.e., an event with incoming and outgoing particles, the particles need to be (somewhat) localized, so we will be considering wavepackets, not plane waves. It also requires the particles to be separated and *not* interacting with each other at early and late times.
We thus depict scattering in a sketch (not a Feynman diagram) like this one:

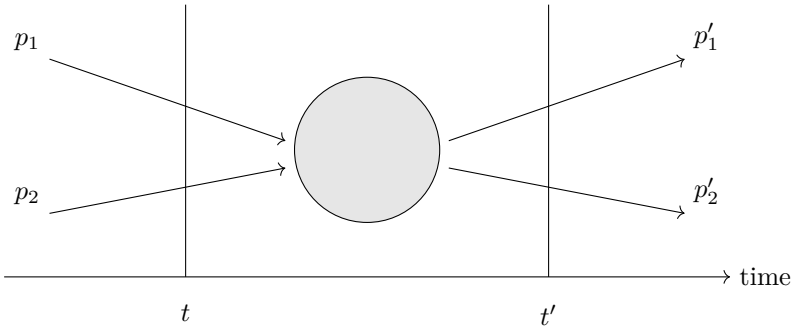


Figure 9: Sketch of 2-2-scattering. The particles interact inside the "blob", but they are initially and finally well-separated, so there is no interaction between them.

To provide the transition amplitude between initial and final state, it *seems* appropriate to define the *scattering-matrix element*:

$$S_{fi} = \left\langle p_1' p_2' \right| T \exp\left(i \int d^4x \, \mathcal{L}_{\text{int}}^I \right) \left| p_1 p_2 \right\rangle . \tag{488}$$

This seems appropriate, because the free single-particle states are time independent in the interaction picture and because our initial and final states contain only free (non-interacting theory) particles.

However, as we will see below, while such a definition is fully correct in QM, it is *not* in QFT. The reason is something that we already encountered twice (in the Casimir effect and in the calculation of the vacuum expectation value of the field from the effective potential) namely that in QFT, even the vacuum is non-trivial. Intuitively speaking, this is due to virtual particles (or, in the path-integral formalism, off-shell field configurations, i.e., those that do not satisfy the classical equations of motion). As a single particle propagates through the vacuum, the interaction term $\lambda\phi^4$ allows it to interact with the virtual particles. Therefore, the single particles that participate in the scattering event always feel the effect of the interaction term $\lambda\phi^4$, even when they are far away from each other and not interact with each other. To properly describe a single particle in a QFT, we will develop the *LSZ formalism* (Lehmann-Symanzik-Zimmermann).

8.3 Källen-Lehmann spectral representation of the propagator

Useful reading: lecture notes on QFT by Timo Weigand; chapters 5 and 13 in Srednicki.

In a non-interacting QFT, we could solve the classical equations of motion, and then write the quantum field in terms of creation and annihilation operators, which are the prefactors of the classical solutions. On this basis, we could construct the full Fock space and understand how the field

operator acts on the vacuum and see that it creates a single (free) particle. This no longer works in an interacting QFT, because step one (finding the classical solutions) in general no longer works.

In short, $\phi(x)|0\rangle$ is not a single-particle state, but $\phi(x)$ also creates multi-particle states.
The first effect of interactions that we will encounter is therefore on the *propagator*, or two-point correlation function (or two-point correlator). We already considered the *two-point-correlator* $\langle 0|\phi(x)\phi(y)|0\rangle$ in q. (168), for the free theory. Now we will investigate what it looks like in the interacting theory. We work in the Heisenberg picture for this. From our considerations of the effective potential, we already know about renormalization, i.e., the actual mass of a particle in the interacting theory is not equal to the mass-parameter in the Lagrangian. To be clear about this difference, we will call the *mass of the free field* m_0. Similarly, there will actually be a change in the normalization of the field, thus we denote the free field by ϕ_0 and the interacting field by ϕ. We had the result that

$$\langle 0|\phi_0(x)\phi_0(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)}, \tag{489}$$

which can be rewritten as

$$\langle 0|\phi_0(x)\phi_0(y)|0\rangle = \int \frac{d^4p}{(2\pi)^3}\, e^{-ip(x-y)}\delta(p^2-m_0^2)\Theta(p^0) = D(x-y;m_0). \tag{490}$$

Here we used the same rewriting that we also used in the derivation of a Lorentz-invariant integration measure in Chapter 3. We are now denoting the propagator as $D(x-y;m_0)$, to make the mass explicit; previously, we just wrote $D(x-y)$ for the same expression.

In the Fourier transform of the free propagator, Eq. (490), we see a delta function contribution at $p^2 = m^2$, i.e., the propagator in Fourier space has a pole at the location of the mass. For the propagator in the free theory, this is all that happens, because the correlation between the value of the field at two points is just related to a single particle propagating from one point to the other.

Now, we are interested in understanding the analogous object in the interacting theory, i.e., $\langle 0|\phi(x)\phi(y)|0\rangle$. Before we evaluate what the right-hand-side looks like, let us consider what we can expect:

- we expect a prefactor compared to the free case that accounts for the change of normalization of the free field
- we expect that there is not just the free-particle pole, but that there is additional structure in the propagator, which accounts for the effect of interactions: first, above $p^2 = 4m^2$, there is sufficient energy available to create multi-particle states. States with two particles exist for all values $p^2 \geqslant 4m^2$, i.e., there is a continuum of states there. Second, in an interacting theory, bound states can form. In a bound state consisting out of two particles, $p^2 < 4m^2$, because the bound-state energy is negative.

We will now investigate the structure of the propagator in a general way that holds irrespective of the specific form of the interaction and without relying on perturbation theory.
We calculate

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\alpha \langle 0|\,\phi(x)\,|\alpha\rangle\langle\alpha|\,\phi(y)\,|0\rangle\,, \tag{491}$$

where we used $\sum_\alpha |\alpha\rangle\langle\alpha| = \mathbb{1}$. The label α labels all states, including multi-particle states (including bound states). We use the label "α" to denote a summation over the continuous parameters characterizing the different states (e.g., momentum), but also discrete parameters (e.g., the number of particles in a bound state). Thus "$\sum_\alpha$" denotes a sum over all such discrete and an integral over all continuous parameters. To bring this into a form which we can most directly compare to the free case, we write

$$\phi(x) = e^{i\hat{P}x}\phi(0)e^{-i\hat{P}x}, \tag{492}$$

where for once we denote the momentum operator $\hat{P}$ with a hat, to make it clear that we are considering an operator. (We have used that $e^{i\hat{P}x}$ generates a spacetime translation, i.e., $\hat{P}$ is the generator of spacetime translations). We also have that $e^{-i\hat{P}x}|\alpha\rangle = e^{-ip_\alpha x}|\alpha\rangle$, i.e., the states $|\alpha\rangle$ are momentum-eigenstates with momentum p_α. Finally, we use that the vacuum is translationally invariant, $e^{i\hat{P}x}|0\rangle = |0\rangle$. Thus

$$\begin{aligned}
\langle 0|\phi(x)\phi(y)|0\rangle &= \sum_\alpha \langle 0|\, e^{i\hat{P}x}\phi(0)e^{-i\hat{P}x}\,|\alpha\rangle\langle\alpha|\, e^{i\hat{P}x}\phi(0)e^{-i\hat{P}y}\,|0\rangle && (493)\\
&= \sum_\alpha \langle 0|\, \phi(0)e^{-ip_\alpha x}\,|\alpha\rangle\langle\alpha|\, e^{ip_\alpha y}\phi(0)\,|0\rangle && (494)\\
&= \sum_\alpha e^{-ip_\alpha(x-y)}|\langle 0|\,\phi(0)\,|\alpha\rangle|^2 && (495)\\
&= \int d^4q\, \sum_\alpha e^{-ip_\alpha(x-y)}\delta^4(q-p_\alpha)|\langle 0|\,\phi(0)\,|\alpha\rangle|^2 && (496)\\
&= \int \frac{d^4q}{(2\pi)^3}\, e^{-iq(x-y)}\rho(q), && (497)
\end{aligned}$$

where we defined

$$\rho(q) = (2\pi)^3 \sum_\alpha \delta^4(q-p_\alpha)|\langle 0|\,\phi(0)\,|\alpha\rangle|^2. \tag{498}$$

Because there are no single- or multi-particle states with negative energy, $q^0 < 0$, we can write

$$\rho(q) = \Theta(q^0)\sigma(q^2), \tag{499}$$

with the *spectral density* $\sigma(q^2)$. It quantifies the contribution of the states $|\alpha\rangle$ to the propagator. We can rewrite a bit further to understand the contribution of $|\alpha\rangle$'s with different masses:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int_0^\infty dM^2 \int \frac{d^4q}{(2\pi)^3}\, e^{-iq(x-y)}\delta(q^2-M^2)\underbrace{\Theta(q^0)\sigma(M^2)}_{\rho(q)}. \tag{500}$$

We rewrote $\sigma(q^2) = \int_0^\infty dM^2\, \delta(q^2-M^2)\sigma(M^2)$. (Note that M^2, not M, is our integration variable, so this is *different* from $\int_0^\infty dM\, \delta(q^2-M^2)\sigma(M^2)$, where we would have extra factors.)
We get back to the free-field case for $\sigma(M^2) = \delta(q^2-m_0^2)$. In general, $\sigma(M^2)$ contains the one-particle contribution with an (as of yet unknown) normalization factor Z,

$$\sigma(q^2) = Z\,\delta(q^2-m^2) + \ldots \qquad (m^2 \neq m_0^2) \tag{501}$$

and further contributions, indicated by the dots. In general we expect the result sketched in Fig. 10.

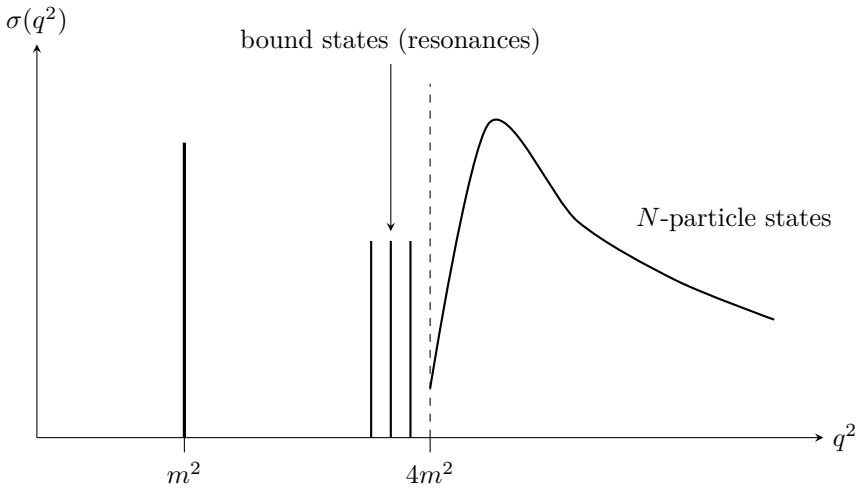


Figure 10: In addition to the single-particle peak at $q^2 = m^2$, there is an N-particle continuum for $q^2 > 4m^2$. Further, just below this point, there are bound states (or resonances), for which the total energy (due to *negative* binding energy) is slightly below $4m^2$. The onset of these bound states is the multiparticle threshold, M_t^2.

Given the structure of $\sigma(q^2)$, it makes sense to split off the single-particle contribution in $\sum_\alpha$ and write

$$\sum_\alpha |\alpha\rangle\langle\alpha| = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,|p\rangle\langle p| + \text{multi-particle} \tag{502}$$

and thus

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\,\underbrace{\langle 0|\,\phi(x)\,|p\rangle}_{=e^{-ipx}\langle 0|\phi(0)|p\rangle}\,\underbrace{\langle p|\,\phi(y)\,|0\rangle}_{=e^{ipy}\langle p|\phi(0)|0\rangle} + \text{multi-particle} \tag{503}$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}} e^{-ip(x-y)}\,\underbrace{|\langle 0|\,\phi(0)\,|p\rangle|^2}_{=:Z} + \dots \tag{504}$$

$$= D(x-y,m)Z + \dots \tag{505}$$

In the last step, we used that Z does not depend on p and can therefore be pulled out of the integral. The fact that it does not depend on p follows from Lorentz invariance:

$$\langle 0|\,\phi(0)\,|p\rangle = \underbrace{\langle 0|\,\Lambda^\dagger}_{\substack{\text{boosted}\\ \text{vacuum}}}\;\phi(0)\;\underbrace{\Lambda\,|p'\rangle}_{\substack{\text{boosted}\\ \text{1-particle}\\ \text{state}}}\,, \tag{506}$$

with $\Lambda^\dagger\phi(0)\Lambda = \phi(0)$, where we used that the scalar field transforms in the trivial representation of the Lorentz group.
Thus,

$$\langle 0|\phi(x)\phi(y)|0\rangle = Z\cdot D(x-y,m^2) + \underbrace{\int_{M_t^2}^\infty dM^2\,\sigma(M^2)D(x-y,M^2)}_{(\star)}\,. \tag{507}$$

For ($\star$), we used

$$\int_{M_t^2}^{\infty} dM^2 \underbrace{\int \frac{d^4q}{(2\pi)^3}\, e^{-iq(x-y)}\delta(q^2-M^2)\Theta(q^0)}_{D(x-y,M^2)}\,\sigma(M^2). \tag{508}$$

This is the *Källen-Lehmann spectral representation* of the correlator. Remember that $\langle 0|\,\phi(x)\phi(y)\,|0\rangle$ is the correlator of the field at two spacetime points and in the free theory corresponds to the probability amplitude for a particle (generated by acting with $\phi(y)$ on the vacuum) to propagate to x. In the interacting theory, the correlation function has another contribution, namely from multi-particle states, i.e., ϕ creates not only 1-particle states.

Thus, we expect that $Z < 1$, because there is an extra contribution from multi-particle states, which is absent in the free theory. Let us show that $Z < 1$ indeed holds:
We consider

$$\begin{aligned}\langle 0|[\phi(x),\phi(y)]|0\rangle &= \langle 0|\phi(x)\phi(y)|0\rangle - \langle 0|\phi(y)\phi(x)|0\rangle &(509)\\ &= Z\big(D(x-y,m^2)-D(y-x,m^2)\big) &(510)\\ &+ \int_{M_t^2}^{\infty} dM^2\,\sigma(M^2)\big(D(x-y,M^2)-D(y-x,M^2)\big). &(511)\end{aligned}$$

We can write this as

$$D(x-y,m^2)-D(y-x,m^2) = \langle 0|[\phi_0(x),\phi_0(y)]|0\rangle\,, \tag{512}$$

because this corresponds to the expectation value of the commutator in the free theory. Because this expression will continue to be important, we denote it by its own letter:

$$\langle 0|[\phi_0(x),\phi_0(y)]|0\rangle = \Delta(x-y,m^2), \tag{513}$$

where we also make the mass explicit. Thus,

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = Z\Delta(x-y,m^2) + \int_{M_t^2}^{\infty} dM^2\,\sigma(M^2)\Delta(x-y,M^2). \tag{514}$$

Note that the mass of Δ under the integral is M^2. Now we would like to use the commutation relation of the free theory, where the subscripts "0" on the fields denote that these are the field operators in the free theory:

$$\big[\phi_0(x^0,\vec{x}),\Pi_0(x^0,\vec{y})\big] = i\delta^3(\vec{x}-\vec{y}) \tag{515}$$

to isolate the factor of Z. We use that $\frac{\partial}{\partial y^0}\Big|_{y^0=x^0}\phi_0(y) = \dot{\phi}_0(x^0,\vec{y}) = \Pi_0(x^0,\vec{y})$.
Thus, we act with $\frac{\partial}{\partial y^0}\Big|_{y^0=x^0}$ on Eq. (514) and obtain

$$\langle 0|\underbrace{\big[\phi(x),\Pi(x^0,\vec{y})\big]}_{\substack{\text{by definition of}\\ \text{our quantization}\\ \text{procedure, this is}\\ \text{also } i\delta^3(\vec{x}-\vec{y})}}|0\rangle = Zi\delta^3(\vec{x}-\vec{y}) + \int_{M_t^2}^{\infty} dM^2\,\sigma(M^2)i\delta^3(\vec{x}-\vec{y}). \tag{516}$$

Now we can remove the $\delta^3(\vec{x}-\vec{y})$ by a volume integral $\int d^3x$ that we apply to both sides. We arrive at the *spectral sum rule*

$$1 = Z + \int_{M_t^2}^{\infty} dM^2\, \sigma(M^2), \tag{517}$$

i.e., $\boxed{Z < 1}$ and $Z = 1$ only if the multi-particle contribution vanishes, which it only does in the free theory. $1 - Z$ accounts for the overlap of $\phi(x)\,|0\rangle$ with multiparticle states.

As we will see in the next section, *time-ordered* correlators are of particular interest to us and thus we write:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = ZD_F(x-y,m^2) + \int_{M_t^2}^{\infty} dM^2\, \sigma(M^2) D_F(x-y,M^2) \tag{518}$$

with the Feynman-propagator

$$D_F(x-y,m_0^2) = \langle 0|T\phi_0(x)\phi_0(y)|0\rangle\,. \tag{519}$$

8.4 S-matrix and asymptotic in/out states

We now consider the scattering of incoming states $|i\rangle$ to outgoing states $|f\rangle$ and want to calculate the transition amplitude. This is formulated in the theory of *asymptotic* in- and out-states:
In-states are created from the asymptotic vacuum $|\text{vac, in}\rangle$, by acting with ϕ_{in} as $t \to -\infty$. We will see that $|\text{vac, in}\rangle$ is the vacuum of the interacting theory, not the free theory, even if the particles in the initial state are well-separated from each other and the interaction is local. Similarly, out-states are created from the asymptotic vacuum $|\text{vac, out}\rangle$ by acting with ϕ_{out} as $t \to +\infty$.

What are ϕ_{in} and ϕ_{out}?
Because the explicit interaction term can be neglected (because all particles are well-separated), but the vacuum is not the vacuum of the free theory, it holds that the mass of the in- an out-fields is not the mass m_0 (the mass of the free theory), but it is m, the mass of the interacting theory. Thus, $\phi_{\text{in}}(x)$ is a free field, obeying the Klein-Gordon-equation with $m \neq m_0$ and can thus be expanded as

$$\phi_{\text{in}}(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\Big(a_{\vec{p},\text{in}}e^{-ipx} + a^{\dagger}_{\vec{p},\text{in}}e^{ipx}\Big), \tag{520}$$

where $p^0 = \sqrt{\vec{p}^2 + m^2}$. Similarly, we can expand ϕ_{out} in term of $a_{\vec{p},\text{out}}$ and $a^{\dagger}_{\vec{p},\text{out}}$.

This allows us to formalize the description of a scattering process, during which

- In the asymptotic past, for $t \to -\infty$, the in-state contains well-separated single particles, which propagate freely as individual states, because the interaction is local. The proper description of these particles is of course not as momentum-eigenstates, but as wavepackets, so that we can indeed ensure that they do not interact and are well-separated.
- At intermediate times, the particles approach each other and the interaction term in the Lagrangian becomes important to describe what happens; the particles *scatter*.
- For $t \to +\infty$, the particles are again well-separated and free (but they may be different particles than in the initial state, e.g., in Quantum Electrodynamics, one can have $e^+e^- \to \mu^+\mu^-$, i.e., an electron and a positron scatter into a muon and anti-muon.)

What is the relation of ϕ_{in} and ϕ_{out} to the interacting field?
Asymptotically, it must hold that

$$\langle\alpha|\,\phi\,|\beta\rangle \to C\,\langle\alpha|\,\phi_{\text{in}}\,|\beta\rangle\,, \tag{521}$$

i.e., at $t \to -\infty$, the matrix elements of the interacting field must agree with those of the in-field (and similarly for $t \to \infty$ and ϕ_{out}). We recall that $|\langle 0|\,\phi(0)\,|p\rangle|^2 = Z$, and state without proof that

$$\langle 0|\,\phi(0)\,|p\rangle = C\,\underbrace{\langle 0|\,\phi_{\text{in}}(0)\,|p\rangle}_{\substack{=1 \text{ by definition} \\ \text{of } \phi_{\text{in}}}} \implies C = \sqrt{Z}. \tag{522}$$

We caution that $\langle\alpha|\,\phi\,|\beta\rangle \to \sqrt{Z}\langle\alpha|\,\phi_{\text{in}}\,|\beta\rangle$ for $t \to -\infty$ is *not* to be taken to mean that one can simply replace ϕ by $\sqrt{Z}\phi_{\text{in}}$ everywhere; in particular, products of $\phi(x)$ cannot simply be replaced by products of ϕ_{in}. In summary, $\phi_{\text{in/out}}$ are free fields whose single-particle states have mass m.

The Hilbert spaces of asymptotic in and out states are isomorphic to each other, which means that

$$|i, \text{in}\rangle = S\,|i, \text{out}\rangle \tag{523}$$

for all states in the in- and out Fock spaces. This S is the S-matrix. It holds that S is unitary.

Mini-Exercise 25. Using $1 = \langle i, \text{in}|i, \text{in}\rangle$ and $\langle i, \text{out}|i, \text{out}\rangle = 1$, show that $S^\dagger = S^{-1}$.

Solution.

$$1 = \langle i, \text{in}|i, \text{in}\rangle = \langle i, \text{out}|\,S^\dagger S\,|i, \text{out}\rangle \quad \text{and} \quad \langle i, \text{out}|i, \text{out}\rangle = 1. \tag{524}$$

The only way to satisfy $\langle i, \text{out}|\,S^\dagger S\,|i, \text{out}\rangle = 1$ and $\langle i, \text{out}|i, \text{out}\rangle = 1$ is by requiring

$$\boxed{S^\dagger S = 1,} \text{ i.e., } \quad \boxed{S^\dagger = S^{-1}.} \tag{525}$$

This means that S is a unitary matrix.

In other words, unitarity of the S-matrix means that we have a well-defined probability interpretation in both in- and out Fock spaces.

Our goal is to compute the transition amplitude

$$\langle f, \text{out}|i, \text{in}\rangle = \underbrace{\langle f, \text{in}|\,S\,|i, \text{in}\rangle}_{\text{S-matrix element}} \tag{526}$$

because $|\langle f, \text{out}|i, \text{in}\rangle|^2$ gives us the probability for scattering from the final to the initial state. Now we re-express $\langle f, \text{in}|\,S\,|i, \text{in}\rangle$ in terms of time-ordered correlators through the LSZ reduction formula.

8.5 Derivation of the LSZ reduction formula

(Useful literature: QFT lecture notes by Timo Weigand; chapter 5 in Srednicki, chapter 6 in Schwartz)

8.5.1 Introduction to LSZ:

The LSZ reduction formula allows us to relate the object that we would like to calculate, namely the S-matrix element, to objects that we will learn how to calculate through Feynman diagrams, namely time-ordered correlation functions. In the chapter on path integrals, we also saw that we can calculate time-ordered correlation functions from the path integral, and Eq. (243) generalizes directly to the interacting case.

To see where we are headed, we state our final result first, and then explain, how to get there: *Lehmann-Symanzik-Zimmermann reduction formula:*

$$\langle p_1, \dots, p_n, \text{out} | q_1, \dots, q_r, \text{in} \rangle = \langle p_1, \dots, p_n, \text{in} | \, S \, | q_1, \dots, q_r, \text{in} \rangle \tag{527}$$

$$= \sum (\text{disconnected terms}) + (iZ^{-\frac{1}{2}})^{n+r} \int d^4y_1 \dots d^4y_n \int d^4x_1 \dots d^4x_r \, e^{i\left(\sum_{k=1}^{n}(p_k y_k) - \sum_{l=1}^{r}(q_l x_l)\right)}$$

$$\cdot (\Box_{y_1} + m^2) \dots (\Box_{x_1} + m^2) \dots \langle 0 | \, T(\phi(y_1) \dots \phi(y_n)\phi(x_1) \dots \phi(x_r)) \, | 0 \rangle \, . \tag{528}$$

This expression tells us that the S-matrix element contains disconnected terms, which do not encode scattering events, but describe situations where a particle simply propagates through from the initial to the final state without taking part in the scattering process. These terms are uninteresting for us. What we are interested in is the additional term on the right-hand side. We note one interesting thing about the right-hand side: For fields that satisfy the free Klein-Gordon equation, $(\Box + m^2)\phi(x) = 0$, the right-hand side would vanish. This happens for the free theory and tells us – not unexpectedly – that there is no particle scattering in the free theory. All physics of particle scattering is encoded in the *deviation* of ϕ from solutions to the free Klein-Gordon equation. Note also that this connects to what we used in discussing the propagator: A field that satisfies the free Klein-Gordon equation can be written in terms of the free creation and annihilation operators, and then $\phi(x)|0\rangle$ generates a one-particle state. The deviation of $\phi(x)$ from a solution of the Klein-Gordon equation, which results in $\phi(x)|0\rangle$ having multi-particle contributions, is exactly what makes the S-matrix element non-zero.

We also note that in Fourier-space, $\Box + m^2 \to -p^2 + m^2$, which goes to zero when a particle goes on shell, i.e., for the asymptotic states. Thus, for the S-matrix to be non-trivial, it must hold that the time-ordered correlator has *poles* whenever a particle goes on-shell. (We already saw the delta-peak for the two-point correlator for the one-particle one-shell condition $p^2 = m^2$.)

In general, it is the case that the correlation functions contain more information about the theory than just the S-matrix. By multipliying the correlation function with $\Box + m^2$ for each of the fields, i.e., with a term that vanishes, we are just picking out the *poles* out of the correlators, which is the only piece of information we need to describe scattering.

8.5.2 Derivation of the LSZ reduction formula

Now let us derive the above expression. We start from the S-matrix elements

$$\langle p_1, \dots, p_n, \text{out} | q_1, \dots, q_r, \text{in} \rangle \, , \tag{529}$$

which are the "building blocks" if we want to describe the scattering of asymptotically localized particles that we describe through localized wave-packets $|f,\text{in}\rangle = \int d^3p\, f(\vec{p})\,|p_\text{in}\rangle$.
To rewrite the S-matrix element in terms of time-ordered correlators, we write

$$|q_i,\text{in}\rangle = a^\dagger_{\vec{q},\text{in}}\,|0\rangle\,, \tag{530}$$

and next rewrite $a^\dagger_{\vec{q},\text{in}}$ in terms of ϕ_in through:

$$\begin{aligned}
a^\dagger_{\vec{q},\text{in}} &= -i\int d^3x\, e^{-iqx}\overset{\leftrightarrow}{\partial_0}\phi_\text{in}(x), \qquad \text{with } A\overset{\leftrightarrow}{\partial_0}B = A\partial_0 B - (\partial_0 A)B && (531)\\
&= -i\int d^3x\, e^{-iqx}\overset{\leftrightarrow}{\partial_0}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\Big(a_{\vec{k},\text{in}}e^{-ikx}+a^\dagger_{\vec{k},\text{in}}e^{+ikx}\Big) && (532)\\
&= -i\int d^3x\, e^{-iqx}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\Big(a_{\vec{k},\text{in}}e^{-ikx}(iq_0-ik_0)+a^\dagger_{\vec{k},\text{in}}e^{ikx}(iq_0+ik_0)\Big) && (533)\\
&= -i\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}}\Big(a_{\vec{k},\text{in}}(iq_0-ik_0)e^{-i(q_0-k_0)x^0}\cdot\delta^3(\vec{k}+\vec{q})+a^\dagger_{\vec{k},\text{in}}e^{-i(q_0+k_0)x^0}(iq_0+ik_0)\cdot\delta^3(\vec{k}-\vec{q})\Big) && (534)\\
&= -i\frac{1}{2\omega_{\vec{q}}}\,(2i\omega_{\vec{q}})\,a^\dagger_{\vec{q},\text{in}}+0 = a^\dagger_{\vec{q},\text{in}}. && (535)
\end{aligned}$$

In the second-to-last step, we used that $\omega_{\vec{q}} = \sqrt{\vec{q}^2+m^2} = \sqrt{(-\vec{q})^2+m^2}$. Using this, we can "trade" $|q_i\rangle$ for $\phi_\text{in}(x)$ as follows:

$$\begin{aligned}
\langle p_1,\ldots,p_n,\text{out}|q_1,\ldots,q_r,\text{in}\rangle &= \langle p_1,\ldots,p_n,\text{out}|\,a^\dagger_{\vec{q}_1,\text{in}}\,|q_2,\ldots,q_r,\text{in}\rangle && (536)\\
&= \frac{1}{i}\int d^3x\, e^{-iq_1x}\overset{\leftrightarrow}{\partial_0}\,\langle p_1,\ldots,p_n,\text{out}|\,\phi_\text{in}(t,\vec{x})\,|q_2,\ldots,q_r,\text{in}\rangle\Big|_{x^0=t}.
\end{aligned}$$

Now we take $t\to-\infty$, because then we can use the relation (cf. Eq. (522) and the discussion that matrix elements of the field are equal (up to a factor of $\sqrt{Z}$) to matrix elements of the in-field in the limit $t\to-\infty$)

$$\lim_{t\to-\infty}\langle\vec{p}|\,\phi_\text{in}(t,\vec{x})\,|0\rangle = \lim_{t\to-\infty}Z^{-\frac{1}{2}}\,\langle\vec{p}|\,\phi(t,\vec{x})\,|0\rangle \tag{537}$$

and thus

$$\langle p_1,\ldots,p_n,\text{out}|q_1,\ldots,q_r,\text{in}\rangle = \lim_{t\to-\infty}Z^{-\frac{1}{2}}\frac{1}{i}\int d^3x\, e^{-iq_1x}\overset{\leftrightarrow}{\partial_0}\langle p_1,\ldots,p_n,\text{out}|\,\phi(x)\,|q_2,\ldots,q_r,\text{in}\rangle\,. \tag{538}$$

$\phi(x)$ can next act on the out-state to annihilate one of the particles in there. To do so, we need to relate the expression for $\lim_{t\to-\infty}$ to a similar expression for $\lim_{t\to+\infty}$, and we do so by using that

$$\left(\lim_{t\to+\infty}-\lim_{t\to-\infty}\right)\int d^3x\, f(t,\vec{x}) = \lim_{\substack{t_f\to+\infty\\ t_i\to-\infty}}\underbrace{\int_{t_i}^{t_f}dt\,\partial_t\int d^3x\, f(t,\vec{x})}_{\int d^4x\,\partial_0 f(t,\vec{x})} \tag{539}$$

for any function $f(t,\vec{x})$.
Applied to our case, this means:

$$\langle p_1,\ldots,p_n,\text{out}|q_1,\ldots,q_r,\text{in}\rangle = \textcircled{1} - \textcircled{2} \tag{540}$$

and

$$\textcircled{1} = \lim_{t\to\infty}(-iZ^{-\frac{1}{2}})\int d^3x\, e^{-iq_1x}\overleftrightarrow{\partial_0}\underbrace{\langle p_1,\dots,p_n,\text{out}|\,\phi(x)\,|q_2,\dots,q_r,\text{in}\rangle}_{\substack{\text{for } t\to+\infty,\text{ this matrix element becomes}\\ \langle p_1,\dots,p_n,\text{out}|\phi_{\text{out}}(x)|q_2,\dots,q_r,\text{in}\rangle}} \tag{541}$$

$$= \langle p_1,\dots,p_n,\text{out}|\,a^\dagger_{\text{out}}(\vec{q}_1)\,|q_2,\dots,q_r,\text{in}\rangle \tag{542}$$

$$= \sum_{k=1}^{n}(2\pi)^3E_{p_k}\delta^3(\vec{p}_k-\vec{q}_1)\,\langle p_1,\dots,\cancel{p_k},\dots,p_n,\text{out}|q_2,\dots q_r,\text{in}\rangle\,, \tag{543}$$

the crossed-out p_k in the last line meaning that p_k has to be removed here. This describes a process, in which one particles does not participate in the scattering event, and thus its in- and outgoing state is identical. While the overall transition amplitude may have such contributions (but note that it is a prerequisite that $p_k = q_1$ for some k - if we can describe a transition in which all p_k are different from all q_i, there is no such contribution because of the delta-function that is there for momentum conservation), they are not interesting to us, because nothing happens to the particle in question.
In terms of Feynman diagrams, this will result in a disconnected part, cf. Fig. 11.

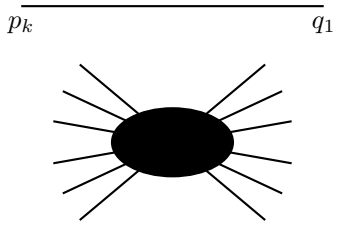


Figure 11: We sketch how one particle just propagates through, whereas the others may participate in an interaction. The interaction region is indicated by the black blob, because we do not yet know how to draw the Feynman diagrams that represent the connected part.

Now let us turn to the contribution $\textcircled{2}$, which describes an actual, non-trivial scattering event.

$$\textcircled{2} = \int d^4x\, Z^{-\frac{1}{2}}\partial_0\Big(e^{-iq_1x}(-i)\overleftrightarrow{\partial_0}\langle p_1,\dots,p_n,\text{out}|\,\phi(x)\,|q_2,\dots,q_r,\text{in}\rangle\Big) \tag{544}$$

$$= \int d^4x\, Z^{-\frac{1}{2}}(-i)\Big((-\partial_0^2e^{-iq_1x})\,\langle p_1,\dots,p_n,\text{out}|\,\phi(x)\,|q_2,\dots,q_r,\text{in}\rangle \tag{545}$$

$$+\,e^{-iq_1x}\partial_0^2\,\langle p_1,\dots,p_n,\text{out}|\,\phi(x)\,|q_2,\dots,q_r,\text{in}\rangle\Big),$$

because the cross-terms cancel each other. Now we can use $-\partial_0^2e^{-iq_1x} = -(i(q_1^0)^2)e^{-iq_1x} = (q_1^0)^2e^{-iq_1x}$. This can be rewritten using that $q_1^2 = m^2$ and thus $(q_1^0)^2 = m^2 + (\vec{q}_1)^2$ and thus

$$-\partial_0^2e^{-iq_1x} = \big(m^2+(\vec{q}_1)^2\big)e^{-iq_1x} = \Big(m^2-(\vec{\nabla})^2\Big)e^{-iq_1x}, \tag{546}$$

where the minus in front of $(\vec{\nabla})^2$ compensates the i^2. Thus,

$$\textcircled{2} = \int d^4x\, Z^{-\frac{1}{2}}(-i)\Bigg(\Big((m^2-\vec{\nabla}^2)e^{-iq_1x}\Big)\,\langle p_1,\dots,p_n,\text{out}|\,\phi(x)\,|q_2,\dots,q_r,\text{in}\rangle \tag{547}$$

$$+\,e^{-iq_1x}\partial_0^2\,\langle p_1,\dots,p_n,\text{out}|\,\phi(x)\,|q_2,\dots,q_r,\text{in}\rangle\Bigg)$$

upon partial integration of $\vec{\nabla}^2$, and using that boundary terms at spatial infinity vanish

$$\textcircled{2} = (-i)Z^{-\frac{1}{2}} \int d^4x_1 \, e^{-iq_1x_1} \big((m^2 + \Box_1) \langle p_1, \dots, p_n, \text{out}| \, \phi(x_1) \, |q_2, \dots, q_r, \text{in}\rangle\big), \tag{548}$$

with $\Box = \partial_0^2 - \vec{\nabla}^2$. As a net result, we have replaced q_1 in $|q_1, \dots, q_r, \text{in}\rangle$ by $\phi(x)$. We now would like to do the same thing with all other particles in the in- and out states, so that we can rewrite the S-matrix element into a correlator $\langle 0| \, \phi(y_1) \dots \phi(y_n)\phi(x_1) \dots \phi(x_r) \, |0\rangle$. To do so, we repeat all steps above for each of the incoming/outgoing particles. There is one point that we need to pay attention to, namely the time-ordering of the field operators. In the exercises, you will check that if one does the derivation properly, one ends up with $T(\phi(y_1)\phi(x_1))$, with the time-ordering operator T.

One can repeat this process for all in- and outgoing particles and obtain the
Lehmann-Symanzik-Zimmermann reduction formula

$$\langle p_1, \dots, p_n, \text{out}|q_1, \dots, q_r, \text{in}\rangle = \langle p_1, \dots, p_n, \text{in}| \, S \, |q_1, \dots, q_r, \text{in}\rangle \tag{549}$$

$$= \sum(\text{disconnected terms}) + (iZ^{-\frac{1}{2}})^{n+r} \int d^4y_1 \dots d^4y_n \int d^4x_1 \dots d^4x_r \, e^{-i\left(\sum_{k=1}^n (-p_k y_k) + \sum_{l=1}^r (q_l x_l)\right)}$$
$$\cdot (\Box_{y_1} + m^2) \dots (\Box_{x_1} + m^2) \dots \langle 0| \, T(\phi(y_1) \dots \phi(y_n)\phi(x_1) \dots \phi(x_r)) \, |0\rangle \, . \tag{550}$$

This expression relates the computation of the S-matrix elements to the computation of time-ordered correlators.

It will be more useful to consider this expression in terms of Fourier transformed quantities.

Mini-Exercise 26. 1) How does

$$(\Box_y + m^2)\phi(y)$$

look like in terms of the Fourier transformed field $\tilde{\phi}(p)$?

2) How does the connected part of the S-matrix element $\langle p_1, \dots, p_n| \, S \, |q_1, \dots, q_r\rangle\big|_{\text{connected}}$ look like in terms of $\langle 0| \, T\tilde{\phi}(p_1) \dots \tilde{\phi}(p_n)\tilde{\phi}(q_1) \dots \tilde{\phi}(q_r)) \, |0\rangle$?

Solution.

$$(\Box_y + m^2)\phi(y) = \int \frac{d^4p}{(2\pi)^4} \, (-p^2 + m^2) e^{-ipy} \tilde{\phi}(p). \tag{551}$$

Thus

$$\langle p_1, \dots, p_n| \, S \, |q_1, \dots, q_r\rangle\Big|_{\text{connected}}$$
$$= (iZ^{-\frac{1}{2}})^{n+r} \prod_{k=1}^n (-p_k^2 + m^2) \prod_{l=1}^r (-q_l^2 + m^2) \cdot \langle 0| \, T\tilde{\phi}(p_1) \dots \tilde{\phi}(p_n)\tilde{\phi}(q_1) \dots \tilde{\phi}(q_r)) \, |0\rangle \, . \tag{552}$$

Note: For all p's and q's it holds that they are on-shell, i.e., $p_1^2 = m^2$ etc.

At a first glance, it therefore looks as if the right-hand-side vanishes and the S-matrix only receives contributions from disconnected parts, i.e., no actual scattering is taking place. Surely this cannot

be correct and of course our conclusion was too hasty: it must be the case that the correlation functions have *poles* when any of the momenta go on-shell, i.e., the pole-structure of the correlator must be $\prod_{k=1}^{n}(-p_k^2+m^2)^{-1}\prod_{l=1}^{r}(-p_l^2+m^2)^{-1}$.

We will indeed find that:

> Correlators have exactly the required pole structure to make S-matrix elements non-zero, but finite,

i.e., there are poles whenever one of the momenta goes on-shell, but they are all simple poles, so that they cancel the factors $(-p^2+m^2)$ etc. in front.

Overall, we can write

$$\boxed{\begin{aligned}&\left(\prod_{k=1}^{n}\frac{i\sqrt{Z}}{p_k^2-m^2}\right)\left(\prod_{l=1}^{r}\frac{i\sqrt{Z}}{p_l^2-m^2}\right)\langle p_1,\dots,p_n|\,S\,|q_1,\dots,q_r\rangle\Big|_{\text{connected}}\\&=\prod_{k=1}^{n}\int d^4y_k\,e^{-ip_ky_k}\prod_{l=1}^{r}\int d^4x_l\,e^{-iq_lx_l}\,\langle 0|\,T(\phi(y_1)\dots\phi(y_n)\phi(x_1)\dots\phi(x_r))\,|0\rangle\,.\end{aligned}} \tag{553}$$

Now we have a concrete prescription for how to calculate S-matrix elements:

- compute the Fourier-transform of the corresponding time-ordered correlation function and take all momenta on-shell
- the result has a contribution with pole-structure $\prod_i \frac{1}{p_i^2-m^2}$, where p_i is the collection of all momenta (ingoing and outgoing)
- the S-matrix element times $(i\sqrt{Z})^{\#\text{particles}}$ is the *residue* with respect to the pole, i.e.,

→ we need to calculate time-ordered correlation functions.

8.6 Calculating time-ordered correlation functions

Correlation functions can be computed in any picture and give the same result. We use this to pick the picture in which the calculation is simplest. This is given if the vacuum state is that of the free theory and the creation/annihilation operators are those of the free theory. We can achieve this in the interaction picture, as we will now see. We will derive:

$$\langle 0|\,T\phi(x_1)\dots\phi(x_n)\,|0\rangle=\frac{\langle 0_{\text{free}}|\,T\phi^I(x_1)\dots\phi^I(x_n)\,e^{-i\int_{-\infty}^{\infty}d\tau\,H_{\text{int}}(\phi^I(\tau,\vec{x}))}\,|0_{\text{free}}\rangle}{\langle 0_{\text{free}}|\,e^{-i\int_{-\infty}^{\infty}d\tau\,H_{\text{int}}(\phi^I(\tau,\vec{x}))}\,|0_{\text{free}}\rangle}. \tag{554}$$

Here, we use $|0_{\text{free}}\rangle$ to denote the vacuum of the free theory. (Many books use the notation that $|0\rangle=|\Omega\rangle$ denotes the vacuum in the interacting theory and $|0_{\text{free}}\rangle=|0\rangle$ denotes the vacuum in the free theory.)

We derive this for the two-point correlator; the derivation generalizes:

$$\langle 0|\,T\phi(x')\phi(x)\,|0\rangle\underset{\substack{\text{choose}\\t'>t}}{=}\langle 0|\,\phi(t',\vec{x}')\phi(t,\vec{x})\,|0\rangle \tag{555}$$

where we choose $t' > t$ without loss of generality. This expression is still entirely in the Heisenberg picture and we now transition to the interacting picture:

$$\langle 0|\, \phi(t', \vec{x}')\phi(t, \vec{x})\, |0\rangle = \langle 0|\, e^{iHt'}\, \phi(0, \vec{x}')e^{-iH(t'-t)}\phi(0, \vec{x})e^{-iHt}\, |0\rangle \tag{556}$$

Here, we have made the transition from the Heisenberg picture to the Schrödinger picture and we now insert $1 = e^{-iH_0t'}e^{iH_0t'}$:

$$= \langle 0|\, \underbrace{e^{iHt'}e^{-iH_0t'}}_{U(0,t')}\, \underbrace{e^{iH_0t'}\phi(0, \vec{x}')e^{-iH_0t'}}_{\phi^I(t',\vec{x}')}\, \underbrace{e^{+iH_0t'}e^{-iH(t'-t)}e^{-iH_0t}}_{U(t',t)}\, \underbrace{e^{iH_0t}\phi(0, \vec{x})e^{-iH_0t}}_{\phi^I(t,\vec{x})}\, \underbrace{e^{iH_0t}e^{-iHt}}_{U(t,0)}\, |0\rangle\,. \tag{557}$$

Thus,

$$\implies \langle 0|\, \phi(t', \vec{x}')\phi(t, \vec{x})\, |0\rangle = \langle 0|\, U(0, t')\, \phi^I(t', \vec{x}')\, U(t', t)\, \phi^I(t, \vec{x})\, U(t, 0)\, |0\rangle\,. \tag{558}$$

Now we have achieved one aspect already, namely that the fields are interaction-picture fields, such that they time-evolve with the *free* Hamiltonian and satisfy the free Klein-Gordon equation and accordingly have an expansion in terms of free creation and annihilation operators.

To relate the interacting vacuum to the free vacuum, we replace the full Hamiltonian

$$H = H_0 + H_{\text{int}} \to H_0 + f(t)H_{\text{int}}, \qquad \text{with } f(t) \to 0 \text{ for } t \to \pm\infty, \tag{559}$$

with $f(t)$ smooth. This amounts to adiabatically switching off the interactions. We also use that we can write

$$U(0, t') = U(0, \infty)U(\infty, t') \tag{560}$$

and therefore have that the vacuum now adiabatically evolves into the *free vacuum* in the limit $t \to \pm\infty$, which we will denote as $|0_{\text{free}}\rangle$, and the $a_{\vec{p}}$ that show up in the mode expansion of ϕ^I actually annihilate $|0_{\text{free}}\rangle$.
Thus we can write

$$U(t, -\infty)U(-\infty, 0)\, |0\rangle = U(t, -\infty)\, |0_{\text{free}}\rangle\langle 0_{\text{free}}|\, U(-\infty, 0)\, |0\rangle\,, \tag{561}$$

where we have inserted a complete set of states and used that the vacuum in the interacting theory evolves into the free vacuum in our setup, so that no other state contributes here. When we considered the path-integral derivation of correlators, we used an alternative trick, namely, instead of inserting the auxiliary function $f(t)$ into the Hamiltonian, we took the limit $t \to \pm\infty(1 - i\varepsilon)$ and thereby removed the overlap with states that are not the vacuum. We could have done the same thing here, and many books do. We are simply learning an alternative to the first way of deriving the result.

Therefore,

$$\langle 0|\, T\phi(t', \vec{x}')\phi(t, \vec{x})\, |0\rangle = \frac{\langle 0_{\text{free}}|\, U(\infty, t')\phi^I(t', \vec{x}')U(t', t)\phi^I(t, \vec{x})U(t, -\infty)\, |0_{\text{free}}\rangle}{\Big(\langle 0|\, U(0, \infty)\, |0_{\text{free}}\rangle\langle 0_{\text{free}}|\, U(-\infty, 0)\, |0\rangle\Big)^{-1}}. \tag{562}$$

We have written the denominator in a curious way, but the reason is that

$$\langle 0|\, U(0,\infty)\, |0_{\text{free}}\rangle^{-1} = \langle 0_{\text{free}}|\, U(\infty,0)\, |0\rangle\,, \tag{563}$$

i.e.,

$$\langle 0|\, U(0,\infty)\, |0_{\text{free}}\rangle \cdot \langle 0_{\text{free}}|\, U(\infty,0)\, |0\rangle = 1 \tag{564}$$

and

$$\Big(\langle 0|\, U(0,\infty)\, |0_{\text{free}}\rangle\langle 0_{\text{free}}|\, U(-\infty,0)\, |0\rangle\Big)^{-1} = \langle 0_{\text{free}}|\, U(\infty,0)\, |0\rangle \cdot \langle 0|\, U(0,-\infty)\, |0_{\text{free}}\rangle \tag{565}$$

$$= \langle 0_{\text{free}}|\, U(\infty,-\infty)\, |0_{\text{free}}\rangle\,. \tag{566}$$

Thus,

$$\langle 0|\, T\phi(t',\vec{x}')\phi(t,\vec{x})\, |0\rangle = \frac{\langle 0_{\text{free}}|\, U(\infty,t')\phi^I(t',\vec{x}')U(t',t)\phi^I(t,\vec{x})U(t,-\infty)\, |0_{\text{free}}\rangle}{\langle 0_{\text{free}}|\, U(\infty,-\infty)\, |0_{\text{free}}\rangle}. \tag{567}$$

(Note on the side: this normalization is why we did not care about normalization factors in the path integral - they drop out.)

Now we use $U(t',t) = e^{-i\int_t^{t'} d\tau\, H^I_{\text{int}}(\tau)}$, which we derived earlier, and write

$$\langle 0|\, T\big(\phi(t',\vec{x}')\phi(x,\vec{x})\, |0\rangle\big) = \tag{568}$$
$$\langle 0_{\text{free}}|\, T\exp\left(-i\int_{t'}^{\infty} d\tau\, H^I_{\text{int}}\right)\phi^I(t')\exp\left(-i\int_t^{t'} d\tau\, H^I_{\text{int}}\right)\phi^I(t)\exp\left(-i\int_{-\infty}^{t} d\tau\, H^I_{\text{int}}\right)|0_{\text{free}}\rangle\,.$$

Under the time-ordering symbol, we can combine all three exponentials into one, because the T arranges all operators into chronological order anyways.
We then obtain a result that we directly write for $n \geqslant 2$:

$$\langle 0|\, T\phi(x_1)\dots\phi(x_n)\, |0\rangle = \frac{\langle 0_{\text{free}}|\, T\phi^I(x_1)\dots\phi^I(x_n)\, e^{-i\int_{-\infty}^{\infty} d\tau\, H_{\text{int}}\left(\phi^I(\tau,\vec{x})\right)}\, |0_{\text{free}}\rangle}{\langle 0_{\text{free}}|\, Te^{-i\int_{-\infty}^{\infty} d\tau\, H^I_{\text{int}}}\, |0_{\text{free}}\rangle}. \tag{569}$$

Note: We have re-expressed the time-ordered correlator that we need for the calculation of S-matrix elements through a correlator in the *free* vacuum and, because we work in the interaction picture, we have fields which can be expanded in terms if *free* creation and annihilation operators.

8.7 Feynman propagator

In the following, we will encounter a particular correlation function, namely the Feynman propagator. We consider it for *free fields in the free theory*. It is closely related to the propagator that we considered in chapter 3, except that it comes with a time-ordering and is defined as

$$D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle. \tag{570}$$

We already encountered it when we considered the Källen-Lehmann spectral representation.
Now, it will be useful to provide a representation for it as an integral over the four-momentum,

namely,

$$D_F(x-y) = \lim_{\varepsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}e^{-ip(x-y)}. \tag{571}$$

The $i\varepsilon$ is introduced so that we can consider the p^0 integration in the complex p^0 plane and perform it using Cauchy's integral theorem. It is there to implement the time-ordering prescription.
To show that this integral representation is correct, we will consider that for $x^0 > y^0$, $D_F(x-y) = D(x-y)$, i.e., the Feynman propagator for this case reduces to the propagator with the same ordering of the times. For $D(x-y)$, we previously had an integral expression, namely

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}e^{-ip(x-y)}. \tag{572}$$

To check that $D_F(x-y)$ as given in Eq. (571) reduces to Eq. (572) (with a given ordering of the times), we check the two cases $x^0 > y^0$ and $x^0 < y^0$ separately. For $x^0 > y^0$, we have that $-ip^0(x^0-y^0) \to -\infty$ for $p^0 \to -i\infty$. Therefore we close the integration contour in the lower half plane as in Fig. 12.

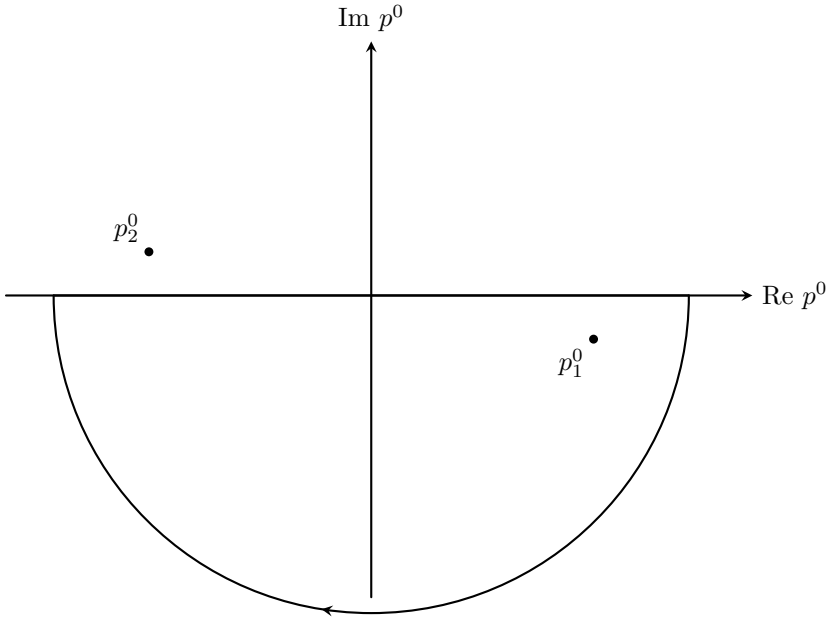


Figure 12: Contour for the integral representation of the Feynman propagator for the case $x^0 > y^0$.

p_1^0 and p_2^0 indicates the position of the two poles, and we have that

$$p^2 - m^2 + i\varepsilon = (p^0 - p_1^0)(p^0 - p_2^0), \tag{573}$$

with $p_{1/2}^0 = \pm\sqrt{\vec{p}^2 + m^2 - i\varepsilon}$. When we rewrite the integral along the contour with imaginary p^0-part, we pick up the residue at the pole:

$$\lim_{p^0\to p_1^0}(p^0-p_1^0)\cdot\frac{i}{(p^0-p_1^0)(p^0-p_2^0)}e^{-ip(x-y)} = \frac{i}{p_1^0-p_2^0}\,e^{-ip(x-y)}\Big|_{p^0=p_1^0} \tag{574}$$

$$= \frac{i}{2\sqrt{\vec{p}^2+m^2-i\varepsilon}}\,e^{-ip(x-y)}\Big|_{p^0=p_1^0}. \tag{575}$$

For $x^0 < y^0$ we proceed analogously, with a contour in the upper half plane, which reduces to the expression for $D(x-y)$ with the corresponding ordering.
Now we are ready to use the Feynman propagator and see how to rewrite the correlation functions, that appeared in the LSZ formalism for the S-matrix, in terms of Feynman diagrams.

8.8 Wick-theorem and Feynman rules

To compute expressions like $\langle 0_{\text{free}}| T\phi^I(x_1)\dots\phi^I(x_n)e^{+i\int d^4x\,\mathcal{L}_{\text{int}}(\phi^I)}|0_{\text{free}}\rangle$ (where we used that $-\int_{-\infty}^{\infty} d\tau\, H_{\text{int}}(\phi^I) = \int d^4x\,\mathcal{L}_{\text{int}}(\phi^I)$), we will

- use perturbation theory, i.e., expand the exponential in powers of λ
- relate the time-ordering to *normal* ordering.

Normal ordering of an operator is defined as writing all the creation operators that appear in an operator to the left and all annihilation operators operators to the right. We denote normal ordering by $:\mathcal{O}:$. For instance, if

$$\mathcal{O} = a_{\vec{p}_1} a^\dagger_{\vec{p}_2} a_{\vec{p}_3} \implies :\mathcal{O}:= a^\dagger_{\vec{p}_2} a_{\vec{p}_1} a_{\vec{p}_3}. \tag{576}$$

The advantage of normal ordering $\phi^I(x_1)\dots\phi^I(x_n)$ is that the expectation value of normal ordered operators vanishes:

$$\boxed{\langle 0|:\mathcal{O}:|0\rangle = 0.} \tag{577}$$

(We use that ϕ^I evolves with the free Hamiltonian and can thus be decomposed into free modes with free creation/annihilation operators.)
Thus, the only contribution to the expectation value that we are interested in will come from the terms that arise due to the rearranging of the time-ordered into a normal-ordered expression.

We start with

$$\langle 0|T\phi(x)\phi(y)|0\rangle, \tag{578}$$

where from now on we drop the I on the field and the subscript on the vacuum, we will even use $\langle T\phi(x)\phi(y)\rangle$ to denote the correlators that we are interested in.
We introduce the shorthand

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}} a_{\vec{p}}\, e^{-ipx} \tag{579}$$

and

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}} a^\dagger_{\vec{p}}\, e^{ipx}, \tag{580}$$

such that $\phi(x) = \phi^+ + \phi^-$ and also $\phi^+|0\rangle = 0 = \langle 0|\,\phi^-$.

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y), & \text{for } x^0 \geqslant y_0 \\ \phi(y)\phi(x), & \text{for } x^0 < y_0. \end{cases} \tag{581}$$

We consider the first possibility first:

$$\phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y). \tag{582}$$

The first, second and last expression are already normal-ordered, but the third one is not, so we have to use the commutator, and

$$\phi(x)\phi(y) =: \phi(x)\phi(y) : + \big[\phi^+(x), \phi^-(y)\big]. \tag{583}$$

We can work out something similar for $\phi(y)\phi(x)$ and obtain

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \Theta(x^0 - y^0)\big[\phi^+(x), \phi^-(y)\big] + \Theta(y^0 - x^0)\big[\phi^+(y), \phi^-(x)\big]. \tag{584}$$

The difference between time-ordered and normal ordered product is a c-number (times the identity operator), because the result of the commutator term is a number. We note that

$$\langle 0|\, T\phi(x)\phi(y)\, |0\rangle = \underbrace{\langle 0|\, :\phi(x)\phi(y): \,|0\rangle}_{=0} + c, \tag{585}$$

where c is the expectation value of the 2nd and 3rd term on the right hand side of the previous expression. Therefore, we have found that the c-number is actually the Feynman propagator, because the definition of the Feynman propagator is exactly that it is $\langle 0|\, T\phi(x)\phi(y)\, |0\rangle$:

$$c = D_F(x - y, m_0^2). \tag{586}$$

We thus define a *contraction*

$$\overbracket{\phi(x)\phi(y)} = D_F(x - y, m_0^2). \tag{587}$$

At this stage, this looks like a somewhat trivial rewriting, but the point is that for time-ordered products of operators, it holds that they can be rewritten into normal-ordered products plus contractions even at the level of the actual operators, and therefore also in expectation values. This is the statement of *Wick's theorem*:

$$T\phi(x_1)\dots\phi(x_n) =: \phi(x_1)\dots\phi(x_n) : + \text{ all contractions of } :\phi(x_1)\dots\phi(x_n): . \tag{588}$$

Here, all contractions signify that we sum over all cases with one or more pairs of fields contracted. Note that this holds as an operator identity (which implies that it also holds in expectation values). For example

$$\begin{aligned} T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)) =& :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): + :\overbracket{\phi(x_1)\phi}(x_2)\phi(x_3)\phi(x_4): \\ &+ :\overbracket{\phi(x_1)\phi(x_2)}\phi(x_3)\phi(x_4): + \text{ other cases with one contracted pair} \\ &+ :\overbracket{\phi(x_1)}\phi(x_2)\overbracket{\phi(x_3)}\phi(x_4): + :\overbracket{\phi(x_1)\phi(x_2)\phi(x_3)}\phi(x_4): \\ &+ :\overbracket{\phi(x_1)\phi(x_2)\phi(x_3)}\phi(x_4): . \end{aligned} \tag{589}$$

In an expectation value, only the fully contracted terms will remain. These can be identified with pairs of Feynman propagators, such that

$$\begin{aligned} \langle 0|\, T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))\, |0\rangle =& D_F(x_1 - x_2, m_0)D_F(x_3 - x_4, m_0) \\ &+ D_F(x_1 - x_3, m_0)D_F(x_2 - x_4, m_0) \\ &+ D_F(x_1 - x_4, m_0)D_F(x_2 - x_3, m_0). \end{aligned} \tag{590}$$

The proof of Wick's theorem proceeds by induction and will be completed in the exercises.

8.9 Feynman rules

Feynman rules associate elements of a Feynman diagram with terms in a calculation. They constitute a pictorial shorthand for the calculation of time-ordered correlators and S-matrix elements.

We represent

$$D_F(x-y) = \quad x \quad y \tag{591}$$

$$(-i\lambda)\int d^4x =$$

and can thus write

$$\begin{aligned}
&\left\langle T\phi(x_1)\dots\phi(x_n)e^{i\int d^4x\,\mathcal{L}_{\text{int}}}\right\rangle \\
&= \overbrace{\phi(x_1)\phi(x_2)\overbrace{\phi(x_3)\phi(x_4)}} + \overbrace{\phi(x_1)\overbrace{\phi(x_2)\phi(x_3)}\phi(x_4)} + \overbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} \\
&\quad + \frac{(-i\lambda)}{4!}\int d^4x\,\overbrace{\phi(x_1)\phi(x)}\overbrace{\phi(x_2)\phi(x)}\overbrace{\phi(x_3)\phi(x)}\overbrace{\phi(x_4)\phi(x)} \\
&\quad + (\text{mini-exercise}) + \text{ terms which are not fully contracted } + \mathcal{O}(\lambda^2).
\end{aligned} \tag{592}$$

In terms of Feynman diagrams this corresponds to

$$\left\langle T\phi(x_1)\dots\phi(x_n)e^{i\int d^4x\,\mathcal{L}_{\text{int}}}\right\rangle$$

$=$ x_1 x_2 x_3 x_4 $+$ x_1 x_3 x_2 x_4 $+$ x_1 x_3 x_2 x_4 $+ \frac{(-i\lambda)}{4!}\int d^4x$ x_1 x_3 x_2 x_4

$+ (\text{mini-exercise}) + \text{ terms which are not fully contracted } + \mathcal{O}(\lambda^2).$ (593)

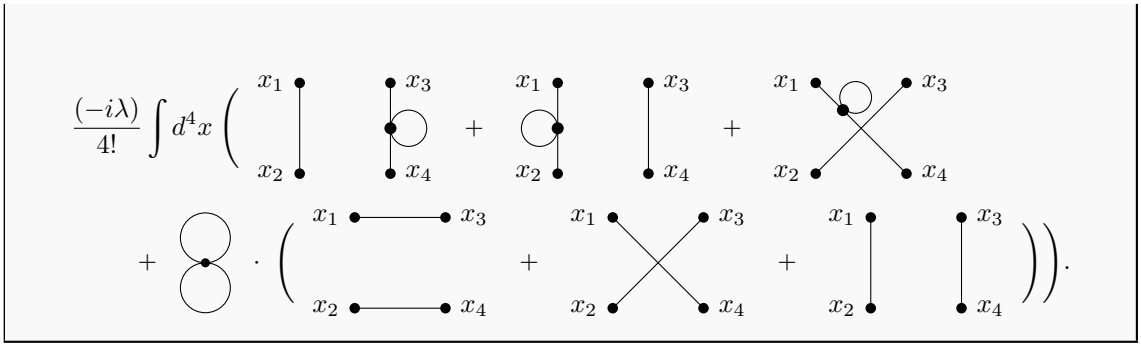
Note the difference between and .

Mini-Exercise 27. What are the additional, fully-contracted expressions $\sim i\lambda$ in the previous expression? Draw the corresponding Feynman diagrams and write the contractions.

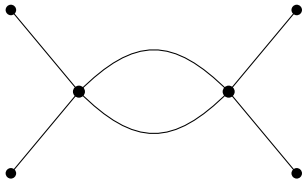
Solution.

$$\begin{aligned}
&\frac{(-i\lambda)}{4!}\int d^4x\,\Big(\overbrace{\phi(x_1)\phi(x_2)}\overbrace{\phi(x)\phi(x)}\overbrace{\phi(x_3)\phi(x)}\overbrace{\phi(x_4)\phi(x)} + \\
&\quad + \overbrace{\phi(x_1)\phi(x)}\overbrace{\phi(x_2)\phi(x)}\overbrace{\phi(x_3)\phi(x_4)}\overbrace{\phi(x)\phi(x)} + \overbrace{\phi(x_1)\phi(x)}\overbrace{\phi(x)\phi(x_4)}\overbrace{\phi(x)\phi(x)}\overbrace{\phi(x_2)\phi(x_2)} \\
&\quad + \overbrace{\phi(x)\phi(x)}\overbrace{\phi(x)\phi(x)}\cdot\Big(\overbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} + \overbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)} + \overbrace{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)}\Big)\Big).
\end{aligned}$$

This corresponds to



We get the first non-trivial loop contribution to 2-2-scattering at $\mathcal{O}(\lambda^2)$, where we have, among others



Note: the diagrams come with non-trivial numerical prefactors that we have not determined in the above.

8.10 Feynman rules in momentum space

Typically, we calculate scattering amplitudes in momentum space, because there they correspond to poles of the correlator. Thus, we write Feynman rules in momentum space:
We use the notation

$$G(x_1, \ldots, x_n) = \langle T(\phi(x_1) \ldots \phi(x_n)) \rangle \tag{594}$$

for the correlators. This is motivated from the fact that the 2-pt function $G(x, y)$ is a Green's function of the Klein-Gordon-operator.
In defining the Fourier transform $\tilde{G}(p_1, \ldots, p_n)$. we choose e^{-ipx} for incoming particles and e^{ipx} for outgoing particles, as is consistent with our discussion for LSZ, so that

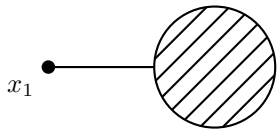
$$\tilde{G}(p_1, \ldots, p_n) = \int d^4x_1 \, e^{-ip_1x_1} \ldots \int d^4x_n \, e^{ip_nx_n} \, G(x_1, \ldots, x_n). \tag{595}$$

Now we start with the real-space Feynman rules:

$$\begin{aligned} x \bullet\!\!-\!\!-\!\!\bullet y &= D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \, \frac{i}{p^2 - m_0^2 + i\varepsilon} e^{-ip(x-y)} \\ \times &= (-i\lambda) \int d^4x \end{aligned} \tag{596}$$

and we need to figure out how to write the corresponding rules in momentum space.
First, we observe that on each external line, the momentum is fixed to the momentum introduced by the Fourier transform, i.e., to an external line starting at some position x_i, the Fourier transform will associate a momentum p_i.

Let's look at a single external line of some diagram to see how this happens:

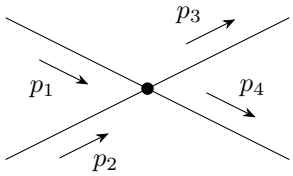


$$\tilde{G}(p_1,\ldots) = \underbrace{\int d^4x_1\, e^{-ip_1x_1}}_{\substack{\text{from Fourier-}\\ \text{transforming } G(x_1,\ldots)}} \underbrace{\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_0^2+i\varepsilon}\,e^{-iq(x_1-y_1)}}_{\substack{\text{from the propagator on this line,}\\ \text{where we assume that the "interior"}\\ \text{point at which the line terminates, is } y_1.}} \tag{597}$$

$$\Longrightarrow \int d^4x_1\, e^{-ip_1x_1}e^{-iqx_1} = \delta^4(p_1+q) \tag{598}$$

$$\Longrightarrow \frac{i}{p_1^2-m_0^2+i\varepsilon}\,e^{-ip_1y_1} \text{ is associated to that line.} \tag{599}$$

What happens to the momenta of the different lines that meet at a vertex?



$$\begin{aligned} &\sim -i\lambda \int d^4x\, e^{-ip_1x}e^{-ip_2x}e^{+ip_3x}e^{+ip_4x} \qquad \left(\text{from the } e^{-ipy_1} \text{ in the example above}\right)\\ &\sim -i\lambda(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4) \end{aligned} \tag{600}$$

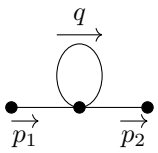
→ this enforces 4-momentum conservation at each vertex. (It is a good cross check of the formalism that a feature such as momentum conservation, which we would expect, shows up explicitly.)
Note that, because each line ends either at an external point or at a vertex, all factors of $e^{\pm ipx}$ are removed. Note also that none of the momenta related to external lines are integrated over, and also, some internal lines have a fixed momentum. However, there are some internal momenta, for which a momentum integration remains and this is referred to as "loop momentum integration", because these internal lines always form some form of loop.

Now that we have seen what happens to a single ingoing line and what happens at a vertex, let's look at a few examples of complete diagrams. We start with the simplest one, which is the propagator:

1)

$$\begin{aligned} &= \int d^4x_1\, e^{-ip_1x_1}\int d^4x_2\, e^{ip_2x_2}D_F(x_2-x_1)\\ &= \int d^4x_1\, e^{-ip_1x_1}\int d^4x_2 e^{+ip_2x_2}\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_0^2+i\varepsilon}e^{-iq(x_2-x_1)}\\ &= \int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_0^2+i\varepsilon}(2\pi)^4\delta^4(p_1-q)(2\pi)^4\delta^4(p_2-q)\\ &= \frac{i}{p_2^2-m_0^2+i\varepsilon}(2\pi)^4\delta^4(p_1-p_2). \end{aligned} \tag{601}$$

2)



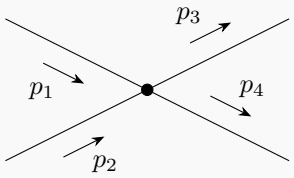
$$
\begin{aligned}
&= \int d^4x_1\, e^{-ip_1x_1} \int d^4x_2\, e^{ip_2x_2}(-i\lambda) \int d^4x\, D_F(x_2-x)D_F(x-x_1)D_F(x-x) \\
&= -i\lambda \int d^4x_1 \int d^4x_2 \int d^4x \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4}\, e^{-ip_1x_1} e^{ip_2x_2} \\
&\quad \cdot \frac{i}{q_1^2 - m_0^2 + i\varepsilon} e^{-iq_1(x_2-x)} \frac{i}{q_2^2 - m_0^2 + i\varepsilon} e^{-iq_2(x-x_1)} \frac{i}{q^2 - m_0^2 + i\varepsilon} \underbrace{e^{-iq(x-x)}}_{=1} \\
&= -i\lambda \int \frac{d^4q}{(2\pi)^4} \left(\frac{i}{p_1^2 - m_0^2 + i\varepsilon} \right)^2 \frac{i}{q^2 - m_0^2 + i\varepsilon} (2\pi)^4 \delta^4(p_1 - p_2).
\end{aligned}
\tag{602}
$$

The "loop-momentum" q is unfixed and integrated over. We may recognize this as the same type of momentum integration as the one that arose in our calculation of the effective potential from the path integral. In fact, the calculation we performed there may be rephrased in terms of Feynman diagrams.

Physically, loop corrections arise due to the non-trivial nature of the vacuum in the interacting quantum theory. We may in fact think of the diagram as the real particle (with momentum p_1), encountering a virtual particle, which, after interacting with the real particle, annihilates again with another virtual particle.[16] Virtual particles exist with *all possible values of the momentum* and thus we have to integrate over all possible momenta to account for the full effect.

3)

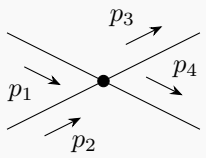
Mini-Exercise 28. Now calculate



(603)

[16] In general, it is always virtual particle-antiparticle pairs that annihilate, because charge must remain conserved. In the case of a real scalar field, there is no distinction between particle and antiparticle, because there is no charge.

Solution.



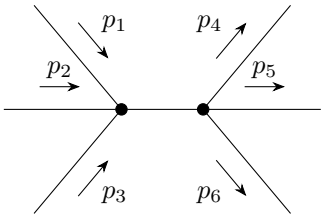
$$= \int d^4x_1\, e^{-ip_1x_1} \int d^4x_2\, e^{-ip_2x_2} \int d^4x_3\, e^{ip_3x_3} \int d^4x_4\, e^{ip_4x_4}$$

$$\cdot (-i\lambda) \int d^4x\, D_F(x-x_1)D_F(x-x_2)D_F(x_3-x)D_F(x_4-x)$$

$$= \frac{i}{p_1^2-m_0^2+i\varepsilon}\frac{i}{p_2^2-m_0^2+i\varepsilon}\frac{i}{p_3^2-m_0^2+i\varepsilon}\frac{i}{p_4^2-m_0^2+i\varepsilon}$$

$$\cdot (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)(-i\lambda).$$

4)



$$= \int d^4x_1\, e^{-ip_1x_1} \int d^4x_2\, e^{-ip_2x_2} \int d^4x_3\, e^{-ip_3x_3}$$

$$\cdot (-i\lambda) \int d^4x\, D_F(x-x_1)D_F(x-x_2)D_F(x-x_3)D_F(x-y) \tag{604}$$

$$\cdot (-i\lambda) \int d^4y\, D_F(y-x_4)D_F(y-x_5)D_F(y-x_6)$$

$$\cdot \int d^4x_4\, e^{ip_4x_4} \int d^4x_5\, e^{ip_5x_5} \int d^4x_6\, e^{ip_6x_6}.$$

We first use the integrals $\int d^4x_i$ associated to the external lines, which results in $\frac{ie^{\pm ip_ix/y}}{p_i^2-m_0^2+i\varepsilon}$ on each external line, depending on whether it is in- or outgoing (and ends at x or y).

$$\to = (-i\lambda)^2 \int d^4x \int d^4y \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m_0^2+i\varepsilon} e^{-ip(x-y)} \tag{605}$$

$$\cdot\, e^{ip_1x}e^{ip_2x}e^{ip_3x}e^{-ip_4y}e^{-ip_5y}e^{-ip_6y} \prod_{i=1}^{6} \frac{i}{p_i^2-m_0^2+i\varepsilon} \tag{606}$$

$$= (-i\lambda)^2\big((2\pi)^4\big)^2 \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m_0^2+i\varepsilon} \tag{607}$$

$$\cdot\, \delta^4(p_1+p_2+p_3-p)\delta^4(p_4+p_5+p_6-p) \prod_{i=1}^{6} \frac{i}{p_i^2-m_0^2+i\varepsilon} \tag{608}$$

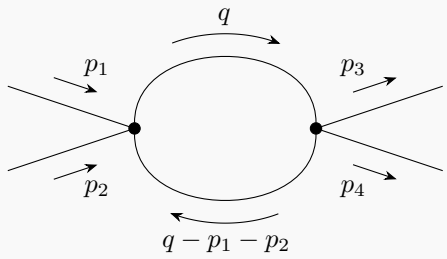
$$= (-i\lambda)^2 \frac{i}{(p_1+p_2+p_3)^2-m_0^2+i\varepsilon} (2\pi)^4\delta^4(p_1+p_2+p_3-p_4-p_5-p_6) \prod_{i=1}^{6} \frac{i}{p_i^2-m_0^2+i\varepsilon} \tag{609}$$

Note: despite there being an internal line, all momenta are fixed, including that of the internal

line. This is because the internal line does not form a loop. The situation is different in our previous example, 3).

5)

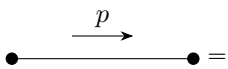
Mini-Exercise 29. Calculate



(610)

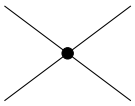
In summary, the momentum-space Feynman rules for $\lambda\phi^4$ theory are:

1)



$$= \frac{i}{p^2 - m_0^2 + i\varepsilon} \tag{611}$$

2)



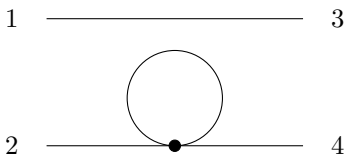
$$= -i\lambda \tag{612}$$

3) assign momenta at each vertex so that momentum conservation is ensured

4) multiply by an overall factor $(2\pi)^4\delta^4(p_f - p_i)$

5) multiply by $\int \frac{d^4p}{(2\pi)^4}$ for each closed loop, with p the momentum that is unfixed after ensuring momentum conservation at each vertex. p is called the "loop momentum".

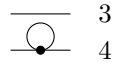
Symmetry factors: some diagrams have additional numerical prefactors. The $\frac{1}{4!}$ in the interaction in the Lagrangian is chosen such that is cancels combinatorial possibilities associated with exchanging the corresponding fields. However, there are diagrams for which non-trivial factors are left. For example:



(613)

If we apply Wick's theorem to $\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4$, we have $4 \cdot 3$ ways of arriving at

$$\overbrace{\phi(x_1)}\phi(x_3)\,\overbrace{\phi(x_2)}^{\substack{\uparrow\\ 4\\ \text{choices}}}\phi\,\overbrace{\phi(x_4)}^{\substack{\uparrow\\ 3\\ \text{choices}}}\phi\,\overbrace{\phi}\phi. \tag{614}$$

This does not fully cancel the $\frac{1}{4!}$ coming from the vertex, so the diagram has a *symmetry factor* $\frac{4\cdot 3}{4!} = \frac{1}{2}$. Intuitively, the symmetry factor is the inverse of the number of ways which the components of a diagram can be interchanged to still give the same diagram. For 

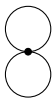
, the two

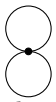
lines that go out from the vertex and close in a loop can be interchanged.

Now we go back to the expression that we started from, namely

$$\frac{\langle 0|T\phi(x_1)\dots\phi(x_n)\,e^{iS_{\rm int}}|0\rangle}{\langle 0|Te^{iS_{\rm int}}|0\rangle}. \tag{615}$$

What is the diagrammatic representation of this expression? $\langle 0|Te^{iS_{\rm int}}|0\rangle$ has a Feynman diagram expansion in terms of *vacuum bubbles*, i.e., diagrams without external lines:

$$\langle 0|Te^{iS_{\rm int}}|0\rangle = 1 + \frac{1}{2}$$  $$+\dots \tag{616}$$

The numerator of this expression also contains the same vacuum bubbles. In fact, if we consider the full Feynman diagram expansion of the numerator, we see that each given Feynman diagram with n external lines appears multiplied with the same factor $1 + \frac{1}{2}$  $+\dots$.

$\Longrightarrow$ The denominator of Eq. (615) *removes* the Feynman diagrams that are just vacuum bubbles, such that

$$\underbrace{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle}_{\text{in Heisenberg picture}} = \frac{\langle 0_{\rm free}|T\phi^I(x_1)\dots\phi^I(x_n)\,e^{iS_{\rm int}}|0_{\rm free}\rangle}{\langle 0_{\rm free}|Te^{iS_{\rm int}}|0_{\rm free}\rangle} = \begin{matrix}\text{sum of all Feynman diagrams}\\ \text{without vacuum bubbles}\end{matrix}. \tag{617}$$

To close this section, let us make a comment on the perturbative series of Feynman diagrams that contributes to any given n-point function:

- First, in many Feynman diagrams with a loop integral, there is an ultraviolet divergence, i.e., the loop integration contains a divergence that comes from the high-momentum part of the integration. This divergence can, in perturbatively renormalizable theories, be understood and treated through perturbative renormalization.
- We see that, as we go to higher orders in the perturbative series in the coupling, the number of Feynman diagrams that contribute, grows very quickly.
- The perturbative series in powers of the coupling is not a convergent series, but only an asymptotic series. Thus, the result of summing the terms in the series (after renormalization) gets closer to the full result, as we add higher-order terms in the coupling, but only up to a finite number of terms, before the result gets *worse*.

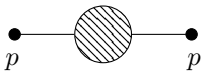
Taken together, the above points mean that calculating observables in interacting QFT is mathematically somewhat more tricky than it is in many other frameworks in physics, where we are neither used to intermediate divergences nor to asymptotic (rather than convergent) series. Nevertheless, QFT makes very precise predictions. For instance, the anomalous magnetic moment of the electron is calculated to twelve significant digits and found to agree with experiment.

8.11 Z-factor and physical mass in Källen-Lehmann spectral representation from Feynman diagrams

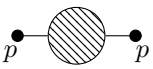
We recall that

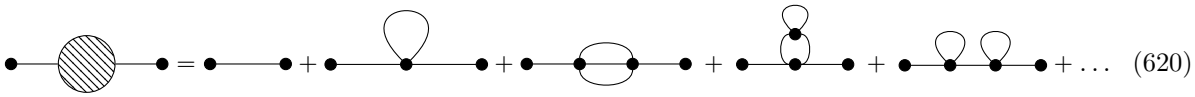
$$\langle 0|T\phi^H(x)\phi^H(y)|0\rangle = Z\, D_F(x-y,m^2) + \int_{M_t^2}^{\infty} dM^2\, \sigma(M^2) D_F(x-y,M^2). \tag{618}$$

We would now like to understand how Z and m^2 are determined in terms of Feynman diagrams. There are divergences, if the computation is done explicitly, but here we will only show how the perturbative expansion can be written as a geometric series.
In momentum space:



$$= \frac{iZ}{p^2-m^2+i\varepsilon} + \int_{M_t^2}^{\infty} dM^2\, \sigma(M^2) \frac{i}{p^2-M^2+i\varepsilon}\,, \tag{619}$$

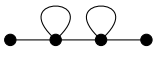
where  denotes the *full* propagator, and is given by a sum over all momentum space Feynman diagrams with two external lines, without vacuum bubbles and without an overall delta-function. From here on, we drop the $i\varepsilon$, because it only matters if we want to perform the p^0-integration and need to know how to close the contour.
We have that

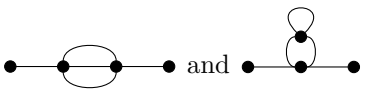


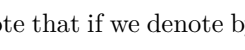
(620)

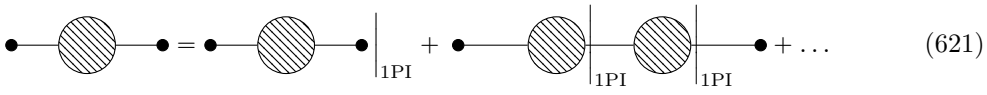
where the first diagram is of order $\mathcal{O}(\lambda^0)$, the second one is of order $\mathcal{O}(\lambda)$ and the last three are of order $\mathcal{O}(\lambda^2)$.

There is an important difference between the diagrams at $\mathcal{O}(\lambda^2)$:

 is called "*1-particle-reducible*", because it falls apart into two disconnected parts when the internal line in the center is cut.

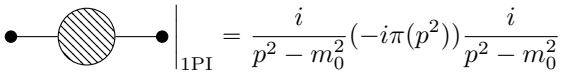
 and are called "*1-particle-irreducible*", because they do *not* fall apart, when any of the internal lines is cut.

We now note that if we denote by  the 1-particle irreducible diagrams (all of them, to any order in λ), where we have removed the external lines (also often called "legs"), then we can write:



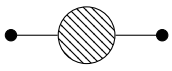
(621)

We denote  $= -i\pi(p^2)$ and it is called the self-energy.
It holds that



$$= \frac{i}{p^2-m_0^2}(-i\pi(p^2))\frac{i}{p^2-m_0^2} \tag{622}$$

and thus we can write, using Eq. (621)

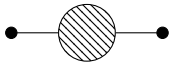


$$= \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}(-i\pi(p^2))\frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}(-i\pi(p^2))\frac{i}{p^2 - m_0^2}(-i\pi(p^2))\frac{i}{p^2 - m_0^2} + \dots \tag{623}$$

Here we see why it was useful to introduce the 1-particle-irreducible set of diagrams: because we did so, we are guaranteed that the momentum that flows through each consecutive propagator that connects the irreducible parts, is p^2.
We now use the series expansion

$$\frac{a}{1-r} = a(1 + r + r^2 + \dots) \tag{624}$$

to write



$$\begin{aligned} &= \frac{i}{p^2 - m_0^2}\left(\frac{i}{1 - (-i\pi(p^2)) \cdot \frac{i}{p^2 - m_0^2}}\right) \\ &= \frac{i}{\cancel{p^2 - m_0^2}} \frac{1}{\frac{i}{\cancel{p^2 - m_0^2}}} \frac{1}{\left(\frac{i}{p^2 - m_0^2}\right)^{-1} + i\pi(p^2)} \\ &= \frac{i}{p^2 - m_0^2 - \pi(p^2)}. \end{aligned} \tag{625}$$

We can equate this with the Källen-Lehmann spectral representation

$$\frac{i}{p^2 - m_0^2 - \pi(p^2)} = \underbrace{\frac{iZ}{p^2 - m^2}}_{\substack{\text{pole at} \\ p^2 = m^2}} + \underbrace{\int_{M_t^2}^{\infty} dM^2\, \sigma(M^2) \frac{i}{p^2 - M^2}}_{\text{no poles at } p^2 = m^2}. \tag{626}$$

$\implies$ the left hand side must also have a pole at $p^2 = m^2$, i.e.,

$$\boxed{p^2 - m_0^2 - \pi(p^2)\Big|_{p^2 = m^2} = 0} \tag{627}$$

and hence $m^2 = m_0^2 + \pi(m^2)$.
In addition, the residues at the pole must match:

$$\lim_{p^2 \to m^2} (p^2 - m^2)\left(\frac{iZ}{p^2 - m^2} + \int_{M_t^2}^{\infty} dM^2\, \sigma(M^2) \frac{i}{p^2 - M^2}\right) = iZ. \tag{628}$$

Thus

$$\lim_{p^2 \to m^2} (p^2 - m^2)\, \frac{i}{p^2 - m_0^2 - \pi(p^2)} \stackrel{!}{=} iZ. \tag{629}$$

To take the limit, we Taylor expand $\pi(p^2 \approx m^2)$ and write

$$Z^{-1} = \lim_{p^2 \to m^2} \frac{p^2 - m_0^2 - \big(\pi(m^2) + \pi'(p^2 - m^2)\big)}{p^2 - m^2} \tag{630}$$

$$= \lim_{p^2 \to m^2} \left(\underbrace{\frac{p^2 - m_0^2 - \pi(m^2)}{p^2 - m^2}}_{=1 \text{ (see above)}} - \pi'(m^2)\right) \tag{631}$$

$$\implies Z^{-1} = 1 - \pi'(m^2). \tag{632}$$

This is a powerful result, because we now understand quantitatively why the mass of 1-particle eigenstate in an interacting theory differs from m_0: The reason lies in the self-interaction of the field (encoded in the Feynman diagrams), which shift the mass from m_0 to m. This justifies our treatment of asymptotic states where we used that they behave like free particles, but with a shifted mass and a non-trivial "wave-function renormalization" Z.

Next, we are ready to come back so S-matrix elements and understand how to write the Feynman rules for them. Then, we derive how the S-matrix is related to scattering cross-sections, such that we understand the relation of Feynman diagrams to scattering cross-sections.

8.12 Feynman rules for scattering amplitudes

Using the notation $G(p_1, \dots, p_n, k_1, \dots, k_m)$ for the time-ordered correlator in Fourier space, the LSZ relation states that

$$\langle p_1, \dots, p_n, \text{out} | k_1, \dots, k_m, \text{in} \rangle \prod_j \frac{i\sqrt{Z}}{p_j^2 - m^2} \prod_i \frac{i\sqrt{Z}}{k_i^2 - m^2} \sim G(p_1, \dots, p_n, k_1, \dots, k_m). \tag{633}$$

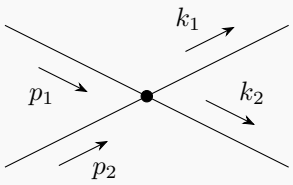
From this, we see that the "amputated" diagrams, for which there is *no* propagator factor on the external legs, form the contributions to the S-matrix element. This is, because these factors, that are part of G, are explicitly divided out in the transition to the S-matrix element. There only remains a factor of $\sqrt{Z}$ for each external line. In addition, we are of course only interested in *connected* diagrams.

In summary, we have the following set of *Feynman rules for the S-matrix*:
To compute $\langle p_1, \dots, p_n | \, S \, | k_1, \dots, k_m \rangle \big|_{\text{connected}}$,

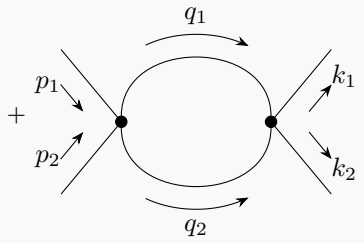
- draw the connected Feynman diagrams with $n + m$ external lines to given order in λ
- assign ingoing momenta k_i and outgoing momenta p_j and label momenta of internal lines with q_i
- each vertex carries $(-i\lambda)(2\pi)^4 \delta^4(\sum \text{ingoing momenta} - \text{outgoing momenta})$. (Using momentum conservation at a vertex may result in a change of some q_i, e.g., $q_i + p_l + k_r$)
- each internal line carries $\frac{i}{q_i^2 - m_0^2 + i\varepsilon}$
- integrate over all internal momenta $\prod_i \int \frac{d^4 q_i}{(2\pi)^4}$
- divide by the symmetry factor
- sum up all diagrams and multiply by $(\sqrt{Z})^{n+m}$.

Mini-Exercise 30. Example: What are the Feynman diagrams contributing to 2-2-scattering to $\mathcal{O}(\lambda^2)$? What are the expressions for them?

Solution.



$$= (-i\lambda)(2\pi)^4\delta^4(p_1+p_2-k_1-k_2)(\sqrt{Z})^{2+2} \tag{634}$$



$$\begin{aligned}
+ &= (-i\lambda)(2\pi)^4 \int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4}\,\delta^4(p_1+p_2-q_1-q_2)\\
&\quad\cdot \frac{i}{q_1^2-m_0^2+i\varepsilon}\frac{i}{q_2^2-m_0^2+i\varepsilon}\\
&\quad\cdot (-i\lambda)(2\pi)^4\delta^4(q_1+q_2-k_1-k_2)(\sqrt{Z})^{2+2}\\
&= (-i\lambda)^2(2\pi)^4\int \frac{d^4q_1}{(2\pi)^4}\frac{i}{q_1^2-m_0^2+i\varepsilon}\\
&\quad\cdot \frac{i}{(q_1-k_1-k_2)^2-m_0^2+i\varepsilon}\delta^4(p_1+p_2-k_1-k_2)(\sqrt{Z})^{2+2}.
\end{aligned} \tag{635}$$

Now we can finally make the connection to cross-sections.

8.13 Cross-section and S-matrix elements

S-matrix elements are generically of the form

$$\langle f|\,S\,|i\rangle = \underbrace{\delta_{fi}}_{\substack{\text{no}\\ \text{scattering}}} + i(2\pi)^4\delta^4(p_f-p_i)\ \underbrace{\mathcal{M}_{fi}}_{\substack{\text{scattering}\\ \text{amplitude}}}\ . \tag{636}$$

The delta-function $\delta^4(p_f-p_i)$ that we have pulled out in general follows by imposing momentum conservation successively at each vertex. The term δ_{fi} is only present if $i=f$, so that there is the possibility of transition from the final to the initial state without any scattering. We exclude this in what follows by assuming $i\neq f$.

The quantum mechanical probability for scattering from an initial state $|i\rangle$ into a final state $|f\rangle$ is given by

$$P_{|i\rangle\to|f\rangle} = |\langle f|\,S\,|i\rangle|^2. \tag{637}$$

If we are considering a *range* of final states $\{|f\rangle\}$ (for instance, later, when we consider Quantum Electrodynamics, we may decide to not fix the polarization of the photons in the final state, then the final state is actually a range of states labeled by the different polarizations that are viable),

then

$$P_{|i\rangle\to|f\rangle} = \sum_{|f\rangle=\{|f\rangle\}} |\langle f|\, S\, |i\rangle|^2 \tag{638}$$

$$= \sum_{|f\rangle=\{|f\rangle\}} \underbrace{\left|(2\pi)^4\delta^4(p_f-p_i)\right|^2}_{=(2\pi)^4\delta^4(p_f-p_i)\cdot\underbrace{(2\pi)^4\cdot\,\delta^4(0)}_{=\mathrm{Vol}(\mathbb{R}^{1,3})}} |\mathcal{M}_{fi}|^2. \tag{639}$$

We see that the overall probability is related to the spacetime volume $\mathrm{Vol}(\mathbb{R}^{1,3})$, thus we define a *transition rate* that is normalized per spacetime volume:

$$\omega_{fi} = \frac{P_{|i\rangle\to\{|f\rangle\}}}{\text{unit time} \times \text{unit spatial volume}}, \tag{640}$$

such that

$$\omega_{fi} = \sum_{f\in\{|f\rangle\}} (2\pi)^4\delta^4(p_f-p_i)\,|\mathcal{M}_{fi}|^2. \tag{641}$$

For scattering into N identical particles, for which the individual momentum are not fixed (just their sum is), $\sum_{f\in\{|f\rangle\}}$ becomes a product of momentum integrals (with Lorentzian normalization) and on-shell,

$$\omega_{fi} = \frac{1}{N!}\prod_{n=1}^{N}\int \frac{d^3k_n}{(2\pi)^3}\,\frac{1}{2E_n}(2\pi)^4\delta^4\left(\sum_i p_i - \sum_n k_n\right)|\mathcal{M}_{fi}|^2, \tag{642}$$

where the $\frac{1}{N!}$ accounts for the indistinguishability of the N identical particles.

Next, we define the cross-section and relate it to ω_{fi}. The *scattering cross-section* σ has units of area and can be understood as the "effective are" of a target that an incoming particle interacts with. It is not the actual area of the target, but this "effective" are - which increases, the stronger the interaction is - that determines the probability of scattering. For instance, the cross-section can be zero, if the interaction between target and particle is zero (e.g., when the target only interacts through the electromagnet interaction, but the particles does *not* interact electromagnetically).

We define σ for the following setup:
We consider an incoming beam of particles of density ρ_B and length l_B, incident upon a target of density ρ_A and length l_A, where $\rho_{A/B}$ is a particles number density.
The number of scattered particles, # events, is

$$\#\text{ events} \sim l_A l_B \int d^2x\;\rho_A(x)\rho_B(x), \tag{643}$$

where we assumed that the number density only varies in the plane orthogonal to the direction of the beam.
The factor of proportionality between # events and $l_A l_B \int d^2x\,\rho_A(x)\rho_B(x)$ has units of area and it is the cross-section:

$$\#\text{ events} = \sigma\, l_A l_B \int d^2x\;\rho_A(x)\rho_B(x). \tag{644}$$

Let's assume that $\rho_A = const$ and $\rho_B = const$ and both beams overlap over an area A, then

$$\sigma = \frac{\#\text{ events}}{l_A l_B \rho_A \rho_B A}. \tag{645}$$

Let's now consider a 2-N-scattering process, for which the initial states are momentum eigenstates $|p_A\rangle$ and $|p_B\rangle$ and the outgoing states are momentum eigenstates $|k_j\rangle$. (More carefully, to describe a localized scattering event, we would actually work with wave-packets here.)
We work in the rest frame of the target, called the *lab frame*, i.e., we consider a setting in which a target is stationary in the lab. We now want to relate σ to ω_{if}, and so we write

$$\frac{\#\text{ events}}{\text{Volume} \times \text{Time}} = \frac{\sigma\, l_A l_B \rho_A \rho_B A}{\text{Volume} \times \text{Time}} \underset{\substack{\uparrow \\ \text{Volume}=l_A A}}{=} \sigma\, \rho_A \rho_B \cdot |\vec{v}_B^{\,\text{L}}|, \tag{646}$$

where Volume $= l_A A$ is the actual interaction volume and $|\vec{v}_B^{\,\text{L}}|$ is the B-velocity in the lab frame, $|\vec{v}_B^{\,\text{L}}| = \frac{l_B}{\text{Time}}$, where the factor "Time" is the time during which scattering occurs.
Because $E = \gamma m$ and $\vec{p} = \gamma m \vec{v}$,

$$|\vec{v}_B^{\,\text{L}}| = \frac{|\vec{p}_B^{\,\text{L}}|}{E_B}. \tag{647}$$

Now we make the transition from # events to *probabilities*. Instead of considering particle number densities, we consider the probability to find particles in a given volume. For this, our (Lorentzian) normalization of the states (see Eq. (132)) is crucial:

$$\langle p_A | p_A \rangle = (2\pi)^3\, 2E_A\, \delta^3(\vec{p}_A - \vec{p}_A) = 2E_A\, V_{\mathbb{R}^3}, \tag{648}$$

where $V_{\mathbb{R}^3}$ is the volume of $\mathbb{R}^3$, i.e. $(2\pi)^3 \delta^3(0) = V_{\mathbb{R}^3}$.
We thus replace

$$\rho_A \to \frac{\langle p_A | p_A \rangle}{V_{\mathbb{R}^3}} = 2E_A \quad \text{and} \quad \rho_B \to 2E_B. \tag{649}$$

In the rest frame of A, in which we are working, we have $2E_A = 2m$, such that

$$\omega_{fi} = \frac{\text{QM probability}}{\text{Volume} \times \text{Time}} \tag{650}$$

$$= \sigma\, \rho_A \rho_B |\vec{v}_B^{\,\text{L}}| \Big|_{\substack{\rho_A = 2E_A = 2m \\ \rho_B = 2E_B}} \tag{651}$$

$$= \sigma\, 2m \cdot 2 \underbrace{E_B |\vec{v}_B^{\,\text{L}}|}_{=|\vec{p}_B^{\,\text{L}}|}. \tag{652}$$

This is derived in the lab frame, but we are interested in providing a Lorentz covariant expression, and thus we rewrite $4m|\vec{p}_B^{\,\text{L}}|$ in such a way that it is invariant under Lorentz boosts in the direction $A \to B$, so that we can also go, e.g., to the center-of-mass frame or to B's rest frame.
It holds that (using that $\vec{v}_A^{\,\text{L}} = 0$)

$$4m|\vec{p}_B^{\,\text{L}}| = 2E_A E_B |\vec{v}_B^{\,\text{L}}| = 4E_A E_B |\vec{v}_A^{\,\text{L}} - \vec{v}_B^{\,\text{L}}|, \tag{653}$$

which is actually invariant under boosts in $A \to B$ direction. Thus,

$$\omega_{fi} = 4E_A E_B |\vec{v}_A - \vec{v}_B| \sigma \tag{654}$$

is the general expression for the transition rate in any frame. (Peskin/Schröder has a more formal proof.)

Now we use our previously derived expression for ω_{fi} in terms of $|\mathcal{M}_{fi}|^2$,

$$\omega_{fi} = \frac{1}{N!}\prod_{n=1}^{N}\int \frac{d^3k_n}{(2\pi)^3}\,\frac{1}{2E_n}(2\pi)^4\delta^4\left(\sum_i p_i - \sum_n k_n\right)|\mathcal{M}_{fi}|^2 \tag{655}$$

for two particles in the initial state to write

$$\boxed{d\sigma = \frac{|\mathcal{M}_{fi}|}{4E_A E_B|\vec{v}_A - \vec{v}_B|}(2\pi)^4\prod_n \frac{d^3k_n}{(2\pi)^3}\,\frac{1}{2E_n}\,\delta^4\Big(p_A + p_B - \sum_n k_n\Big),} \tag{656}$$

which is the *differential* cross section.

9 QED processes at tree level

We now derive the Feynman rules for QED and calculate the cross-section (at tree level, i.e., where the Feynman diagram is "tree-like", i.e., has no loops) for some scattering processes. To do so, we need the propagators for fermions and for photons, as well as the Feynman rules associated to external particles and vertices.

9.1 Feynman rules for QED

We start from the QED Lagrangian

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 + \overline{\psi}\big(i\partial\!\!\!/ - m_0\big)\psi - eA_\mu\overline{\psi}\gamma^\mu\psi. \tag{657}$$

Herein, we have already introduced a gauge-fixing term with gauge parameter λ. It is an important property of a gauge theory that physical results cannot depend on the gauge parameter (as long as one has chosen an admissible gauge condition) and thus one can cross-check calculations by checking that any dependence on λ drops out in physical processes.

9.1.1 Photon propagator

To derive the photon propagator, we take advantage of the path integral for gauge fields, where

$$Z[j] = \int \mathcal{D}A_\mu \, e^{iS + i\int d^4x \, j_\mu A^\mu}, \tag{658}$$

and

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2\right). \tag{659}$$

When calculating the photon propagator, we can ignore the fermion fields, which is why we have not written them out above.
We first perform the A^μ-integral in $Z[j]$, because the exponential is quadratic in A^μ and so we can do the integral by "completing the square" and doing the Gaussian integral. From the resulting expression, we can take derivations with respect to j^μ and use that, just like for the scalar field,

$$\langle T A_\chi(x) A_\lambda(y)\rangle = \frac{(-i)^2}{Z[0]}\frac{\delta}{\delta j^\chi(x)}\frac{\delta}{\delta j^\lambda(y)}\, Z[j]\bigg|_{j=0} \tag{660}$$

where we are dividing by $Z[0]$ to take care f the fact that $Z[j]$ is not properly normalized.
We use

$$\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2\right) = \int d^4x \left(-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{\lambda}{2}\partial_\mu A^\mu \partial_\nu A^\nu\right) \tag{661}$$

$$= \int d^4x \, A_\mu \left(+\frac{1}{2}\partial^2\eta^{\mu\nu} - \frac{(1-\lambda)}{2}\partial^\mu\partial^\nu\right)A_\nu \tag{662}$$

This implies that

$$\implies iS + i\int d^4x \, j_\mu A^\mu = i\int d^4x \left(\frac{A_\mu}{2}\underbrace{\left[\partial^2\eta^{\mu\nu} - (1-\lambda)\partial^\mu\partial^\nu\right]}_{=:(G^{-1})^{\mu\nu}} A_\nu + j_\mu A^\mu\right) \tag{663}$$

$$= i \int d^4x \left(\frac{1}{2}(A_\mu + (G_{\mu\chi} J^\chi)) \big(G^{-1}\big)^{\mu\nu} \big(A_\nu + G_{\nu\lambda} j^\lambda\big) - \frac{1}{2} J_\mu G^{\mu\nu} j_\nu \right). \tag{664}$$

Next, we shift the integration variable in the path integral, $A_\mu + G_{\mu\chi} j^\chi \to A_\mu$ and have

$$Z[j] = \int \mathcal{D}A_\mu \, e^{i \int d^4x \, \frac{1}{2} A_\mu (G^{-1})^{\mu\nu} A_\nu - \frac{i}{2} \int d^4x \, j_\mu G^{\mu\nu} j_\nu}. \tag{665}$$

We can now perform the A integral, but don't even have to in order to calculate $\langle T A_\chi(x) A_\lambda(y) \rangle$, because the part dependent on j_μ n longer depends on A_μ and can be pulled out of the integral. Thus

$$\langle T A_\chi(x) A_\lambda(y) \rangle = \frac{(-i)^2}{Z[0]} \frac{\delta}{\delta j^\chi(x)} \frac{\delta}{\delta j^\lambda(y)} \, Z[j] \Big|_{j=0} \tag{666}$$

$$= (-i)^2 \frac{\delta}{\delta j^\chi(x)} \frac{\delta}{\delta j^\lambda(y)} \, e^{-\frac{i}{2} \int d^4z \, j_\mu(z) G^{\mu\nu} j_\nu(z)} \Big|_{j=0} \tag{667}$$

$$= (-i)^2 \left(\frac{-i}{2} \right) \cdot 2 \, G^{\chi\lambda}(x-y) \tag{668}$$

$$= i \big(\partial^2 \eta^{\chi\lambda} - (1-\lambda) \partial^\chi \partial^\lambda \big)^{-1}. \tag{669}$$

It remains for us to invert the matrix $\partial^2 \eta^{\chi\lambda} - (1-\lambda)\partial^\chi \partial^\lambda$. Because we will be using the Feynman rules in momentum space, we rewrite this matrix in momentum space, where each ∂^μ is substituted by an ip^μ. (Alternatively, we can consider the correlator directly in momentum space and write $\big\langle T \tilde{A}^\chi(p) \tilde{A}^\lambda(-p) \big\rangle$ and Fourier transform A_μ and j_μ in the path integral.)

We have

$$G^{-1}_{\mu\nu}(p) = i \big(p^2 \eta_{\mu\nu} - p_\mu p_\nu (1-\lambda) \big). \tag{670}$$

To find the inverse $G_{\mu\nu}(p)$ explicitly, we make an ansatz

$$G_{\mu\nu}(p) = \frac{1}{p^2} (a \, \eta_{\mu\nu} + b \, p_\mu p_\nu), \tag{671}$$

with a and b constant that are to be determined. Because $\eta_{\mu\nu}$ and $p_\mu p_\nu$ are the only Lorentz-covariant 2-tensors we have available, we include them in our ansatz.
We require

$$(G^{-1})_{\mu\nu} G^{\nu\chi} = \delta_\mu{}^\chi \tag{672}$$

and have

$$i \big(p^2 \eta_{\mu\nu} - p_\mu p_\nu (1-\lambda) \big) \frac{1}{p^2} (a \, \eta^{\nu\chi} + b \, p^\nu p^\chi) \stackrel{!}{=} \delta_\mu{}^\chi \tag{673}$$

$$\iff \underbrace{i p^2 \eta_{\mu\nu} \frac{1}{p^2} a \eta^{\nu\chi}}_{a \delta_\mu{}^\chi} \underbrace{- i p_\mu p_\nu (1-\lambda) \frac{1}{p^2} p^\nu p^\chi b}_{i p_\mu p^\chi (1-\lambda) b} \underbrace{- i p_\mu p_\nu (1-\lambda) \frac{i}{p^2} a \eta^{\nu\chi}}_{i \frac{p_\mu p^\chi}{p^2} (1-\lambda) a} + \underbrace{i p^2 \eta_{\mu\nu} \frac{1}{p^2} b p^\nu p^\chi}_{i b p_\mu p^\chi} \stackrel{!}{=} \delta_\mu{}^\chi \tag{674}$$

$$\implies a = -i \tag{675}$$

$$-\not{\ell}(1-\lambda)b - \not{\ell}(1-\lambda)\frac{a}{p^2} + \not{\ell} b \stackrel{!}{=} 0 \tag{676}$$

$$b(1-(1-\lambda)) = (1-\lambda)\frac{(-i)}{p^2} \qquad (\text{using } a = -i) \tag{677}$$

$$b = \frac{1-\lambda}{\lambda}\frac{(-i)}{p^2} \tag{678}$$

$$\implies G_{\mu\nu}(p^2) = \frac{-i}{p^2}\left(\eta_{\mu\nu} + \frac{(1-\lambda)}{\lambda}\frac{p_\mu p_\nu}{p^2}\right) = \langle T A_\mu(p) A_\nu(-p)\rangle\,. \tag{679}$$

For the Feynman propagator, we of course have to account for the $i\varepsilon$-prescription, which we did not do above, so we have

$$G^F_{\mu\nu}(p^2) = \frac{-i}{p^2 + i\varepsilon}\left(\eta_{\mu\nu} + \frac{(1-\lambda)}{\lambda}\frac{p_\mu p_\nu}{p^2}\right). \tag{680}$$

We see that Feynman gauge ($\lambda = 1$) is a particularly simple choice.

9.1.2 Fermion propagator

The Feynman propagator for fermions is

$$S_F(x-y) = \left\langle T\psi(x)\overline{\psi}(y)\right\rangle, \tag{681}$$

while $\langle T\psi(x)\psi(y)\rangle = 0 = \left\langle T\overline{\psi}(x)\overline{\psi}(y)\right\rangle$. We can evaluate, given

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\Big(a_s(\vec{p})u_s(\vec{p})e^{-ipx} + b_s^\dagger(\vec{p})v_s(\vec{p})e^{ipx}\Big) \tag{682}$$

$$\overline{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\Big(b_s(\vec{p})\bar{v}_s(\vec{p})e^{-ipx} + a_s^\dagger(\vec{p})\bar{u}_s(\vec{p})e^{ipx}\Big), \tag{683}$$

$$S_F(x-y) = \Theta(x^0-y^0)\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\big(\not{p} + m_0\big)e^{-ip(x-y)} \tag{684}$$

$$- \Theta(y^0-x^0)\int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{p}}}\big(\not{p} - m_0\big)e^{ip(x-y)} \tag{685}$$

$$= \big(i\not{\partial}_x - m_0\big)\;\underbrace{D_F(x-y,m_0)}_{\substack{\text{Feynman propagator}\\ \text{of the scalar theory}}} \tag{686}$$

In this derivation, we used the relations $\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m_0$ and $\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m_0$.
$\implies$ We already know the representation as an integral over $\int d^4p$ as well as the representation in momentum space!
We have

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon}\,e^{-ip(x-y)} \tag{687}$$

and in momentum space,

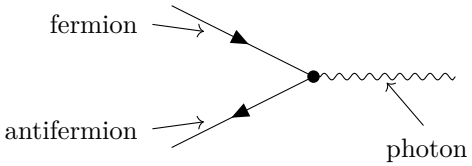
$$S_F(p) = \frac{i(\not{p} + m_0)}{p^2 - m_0^2 + i\varepsilon}. \tag{688}$$

9.1.3 Feynman rules for QED in momentum space

To compute the scattering amplitude $i\mathcal{M}_{fi}$ for a given process, we draw all corresponding, fully connected, amputated Feynman diagrams to the given order in the coupling constant e and read

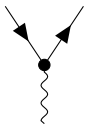
off the expression for $i\mathcal{M}_{fi}$ by:

- each interaction vertex has the form



  and carries a factor $-ie\gamma^\mu$.

  Note: the vertex can also be drawn like this


, (689)

  so that both fermionic lines point in the same direction. The arrow follows the flow of the charge, with an arrow pointing to the right denoting a fermion, and an arrow pointing to the left denoting an antifermion, with opposite charge.

- each internal photon line carries a factor

$$\frac{-i\eta^{\mu\nu}}{p^2+i\varepsilon} \qquad \text{(when we work in Feynman gauge)} \tag{690}$$

  and is drawn as $\mu \sim\sim\sim \nu$

- each internal fermion line is drawn as $\longrightarrow$ and carries a factor

$$\frac{i(\not{p}+m_0)}{p^2-m_0^2+i\varepsilon}. \tag{691}$$

- we impose momentum conservation at each vertex.
- we integrate over all remaining, undetermined, internal momenta with a factor $\int \frac{d^4p}{(2\pi)^4}$
- each ingoing photon of polarization λ carries a factor

$$\epsilon^{\mu\,(\lambda)}(\vec{p})\, Z_A^{1/2}, \tag{692}$$

  each outgoing photon a factor $\epsilon^{\mu\,(\lambda)}(\vec{p})\, Z_A^{1/2}$.

  $\Big($We allow for complex polarization vectors to describe circular polarization,

$$\epsilon^{\mu\,\pm}(\vec{p}) = \pm\frac{1}{\sqrt{2}}\Big(\epsilon^{\mu\,(1)}(\vec{p}) \pm i\epsilon^{\mu\,(2)}(\vec{p})\Big).\Big) \tag{693}$$

- each ingoing fermion with spin s carries a factor $u_s(\vec{p})Z_e^{1/2}$,
- each ingoing antifermion with spin s carries a factor $\overline{v}_s(\vec{p})Z_e^{1/2}$,

- each outgoing fermion with spin s carries a factor $\overline{u}_s(\vec{p})Z_e^{1/2}$,
- each outgoing antifermion with spin s carries a factor $v_s(\vec{p})Z_e^{1/2}$.
- Diagrams may carry overall *signs* due to the anticommuting nature of fermions.

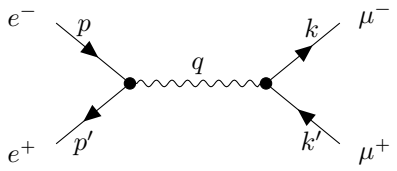
The association of fermions with the u-spinors and antifermions with the v-spinors comes from the consideration of the electric charge, i.e., the conserved charge associated with the $U(1)$ symmetry of the QED Lagrangian. We have that the particles created by $a^\dagger$ are fermions and the particles created by $b^\dagger$ are the antifermions, because they create particles with opposite charge. Through the LSZ-formalism, the a-operators are associated with the u-spinors and the b-operators with the v-spinors. The propagator shows an example that we associate the outgoing particles with $\bar{\psi}$ and the ingoing ones with ψ, which explains the association of ingoing and outgoing with barred and unbarred u's and v's.
Example: Electron-electron scattering, i.e., $e^-e^- \to e^-e^-$:

Mini-Exercise 31. What are the corresponding two tree-level diagrams (i.e., diagrams without internal loops)?

Mini-Exercise 32. What is the expression for the amplitude?
Note the following rule: If two diagrams are related to each other by the crossing of a fermion line, they have a relative minus sign. (The overall sign drops out of the cross-section, which only depends on $|i\mathcal{M}_{fi}|^2$, so it is enough to know the relative sign.)

9.2 Cross-section for $e^+e^- \to \mu^+\mu^-$

As an example for the full calculation of a tree-level diagram in QED, we consider $e^+e^- \to \mu^+\mu^-$. There is only a single Feynman diagram that contributes, namely



(694)

$$i\mathcal{M} = (ie)^2 \overline{v}_{s'}(p')\gamma^\mu u_s(p) \frac{(-i\eta_{\mu\nu})}{q^2 + i\varepsilon} \overline{u}_r(k)\gamma^\nu v_{r'}(k'), \tag{695}$$

where $q = p + p' = k + k'$. Thus

$$i\mathcal{M} = \frac{ie^2}{q^2} \overline{v}_{s'}(p')\gamma^\mu u_s(p) \overline{u}_r(k)\gamma_\mu v_{r'}(k'). \tag{696}$$

Next, we have to decide whether or not we are interested in a situation where the external spins are fixed. Often, one considers an *unpolarized* initial state, so that one averages over all spins in the initial state. Similarly, if one is not interested in distinguishing different values of the spin in

the final state, one *sums* over all spins in the final state and thus considers

$$\underbrace{\frac{1}{2}\sum_{s}}_{\text{average}}\frac{1}{2}\sum_{s'}\sum_{r}\sum_{r'}|i\mathcal{M}|^2. \tag{697}$$

Then we can write

$$\begin{aligned}\frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}\sum_{r}\sum_{r'}|i\mathcal{M}|^2 = \frac{1}{4}\sum_{\substack{s,s'\\r,r'}}\frac{ie^2}{q^2}\cdot\frac{(-ie^2)}{q^2}\big(\overline{v}_{s'}(p')\gamma^\mu u_s(p)\big)\big(\overline{u}_r(k)\gamma_\mu v_{r'}(k')\big)\\ \cdot\big(\overline{v}_{s'}(p')\gamma^\nu u_s(p)\big)^*\big(\overline{u}_r(k)\gamma_\nu v_{r'}(k')\big)^*.\end{aligned} \tag{698}$$

and use that (for $(\gamma^\nu)^\dagger\big(\gamma^0\big)^\dagger = \gamma^0\gamma^\nu$ cf. discussion above Eq. (375))

$$(\overline{v}_{s'}\gamma^\nu u_s)^* = u_s^\dagger\underbrace{(\gamma^\nu)^\dagger\big(\gamma^0\big)^\dagger}_{=\gamma^0\gamma^\nu}v_{s'} = \overline{u}_s\gamma^\nu v_{s'}. \tag{699}$$

We then have

$$\begin{aligned}&\frac{1}{2}\sum_{s}\frac{1}{2}\sum_{s'}\sum_{r}\sum_{r'}|i\mathcal{M}|^2\\ =&\frac{1}{4}\sum_{\substack{s,s'\\r,r'}}\frac{e^4}{q^4}\underbrace{\big(\overline{v}_{s'}(p')\gamma^\mu u_s(p)\big)}_{①}\underbrace{\big(\overline{u}_r(k)\gamma_\mu v_{r'}(k')\big)}_{②}\underbrace{\big(\overline{u}_s(p)\gamma^\nu v_{s'}(p')\big)}_{③}\underbrace{\big(\overline{v}_{r'}(k')\gamma_\nu u_r(k)\big)}_{④}.\end{aligned} \tag{700}$$

In the next step, we will make use of the completeness relation

$$\sum_{s}u_s^a(p)\,\overline{u}_s^b(p) = \not{p}^{ab} + m_e\mathbb{1}^{ab} \tag{701}$$

$$\sum_{s}v_s^a(p)\,\overline{v}_s^b(p) = \not{p}^{ab} - m_e\mathbb{1}^{ab}, \tag{702}$$

where m_e is the electron mass, because u_r and v_s denote e^- and e^+. To do so, it is useful to make the Dirac indices explicit in $|i\mathcal{M}|^2$. We note that the terms within the brackets ( ) always have fully contracted indices. Thus, if we, e.g., pick up all spinors with spin value s, we have

$$\sum_{s}\Big(\overline{v}_{s'}^a(p')(\gamma^\mu)^{ab}u_s^b(p)\Big)\Big(\overline{u}_s^c(p)(\gamma^\nu)^{cd}v_{s'}^d(p')\Big) = \big(\not{p}^{bc} + m\mathbb{1}^{bc}\big)\cdot(\gamma^\mu)^{ab}\,\overline{v}_{s'}^a(p')(\gamma^\nu)^{cd}v_{s'}^d(p') \tag{703}$$

and thus

$$\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{q^4}\,\mathrm{tr}\underbrace{\Big((\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\Big)}_{\text{from term ① and ③}}\cdot\,\mathrm{tr}\underbrace{\Big((\not{k}' + m_\mu)\gamma_\mu(\not{k} - m_\mu)\gamma_\nu\Big)}_{\text{from term ② and ④}}, \tag{704}$$

where the trace is in Dirac indices and we denote the electron mass by m_e and the muon-mass by m_μ.

We are now left with a trace over Dirac indices in gamma-matrices, for which there is a list of *trace identities* that can be proven. We provide the full list here, because this "trace technology"

is relevant for all diagrams in QED (given that there are always gamma matrices on the vertices and also on the internal fermion propagator). (See, e.g., ch. 47 in Srednicki for literature on this.)

- $\operatorname{tr}\gamma^\mu = 0$, because

$$\operatorname{tr}\gamma^\mu = \operatorname{tr}\underbrace{\left(\gamma^5\right)^2}_{=1}\gamma^\mu = -\operatorname{tr}\gamma^5\gamma^\mu\gamma^5 = -\operatorname{tr}\left(\gamma^5\right)^2\gamma^\mu \implies \operatorname{tr}\gamma^\mu = 0, \tag{705}$$

  where we used $\left\{\gamma^\mu, \gamma^5\right\} = 0$ in the second step and the cyclicity of the trace in the third step:

$$\operatorname{tr}\overbrace{\gamma^5\,\gamma^\mu\gamma^5} = \operatorname{tr}\left(\gamma^5\right)^2\gamma^\mu \qquad \text{(cyclicity)} \tag{706}$$

- $\operatorname{tr}\gamma^{\mu_1}\ldots\gamma^{\mu_n} = 0$ for n odd, because

$$\operatorname{tr}\gamma^{\mu_1}\ldots\gamma^{\mu_n} = \operatorname{tr}\left(\gamma^5\right)^2\gamma^{\mu_1}\ldots\gamma^{\mu_n} = -\operatorname{tr}\gamma^5\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma^5 \tag{707}$$

$$= -\operatorname{tr}\left(\gamma^5\right)^2\gamma^{\mu_1}\ldots\gamma^{\mu_n} = -\operatorname{tr}\gamma^{\mu_1}\ldots\gamma^{\mu_n}, \tag{708}$$

  where we again used $\left\{\gamma^\mu, \gamma^5\right\} = 0$ in the second and the cyclicity of the trace in the third step.

- $\operatorname{tr}\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}$, because

$$\operatorname{tr}\gamma^\mu\gamma^\nu = \operatorname{tr}\left(2\eta^{\mu\nu}\,\mathbb{1} - \gamma^\mu\gamma^\nu\right) = 2\eta^{\mu\nu}\underbrace{\operatorname{tr}\mathbb{1}}_{=4} - \underbrace{\operatorname{tr}\gamma^\nu\gamma^\mu}_{\substack{=\operatorname{tr}\gamma^\mu\gamma^\nu \\ \text{(cyclicity)}}} \tag{709}$$

- $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$, because

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \operatorname{tr}\left(\{\gamma^\mu,\gamma^\nu\}\gamma^\rho\gamma^\sigma - \gamma^\nu\{\gamma^\mu,\gamma^\rho\}\gamma^\sigma + \gamma^\nu\gamma^\rho\{\gamma^\mu,\gamma^\sigma\} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right) \tag{710}$$

$$= \operatorname{tr}2\eta^{\mu\nu}\mathbb{1}\gamma^\rho\gamma^\sigma - \operatorname{tr}\gamma^\nu 2\eta^{\mu\rho}\mathbb{1}\gamma^\sigma + \gamma^\nu\gamma^\rho 2\eta^{\mu\sigma}\mathbb{1} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \tag{711}$$

  where we used the cyclicity of the trace in the last term, and where

$$\operatorname{tr}2\eta^{\mu\nu}\mathbb{1}\gamma^\rho\gamma^\sigma = 2\eta^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma = 8\eta^{\mu\nu}\eta^{\rho\sigma} \tag{712}$$

  etc...

- $\operatorname{tr}\gamma^5 = 0$, because

$$\operatorname{tr}\gamma^5 = \operatorname{tr}\gamma^5\underbrace{\left(\gamma^0\right)^2}_{=1} = -\operatorname{tr}\gamma^0\gamma^5\gamma^0 = -\operatorname{tr}\underbrace{\left(\gamma^0\right)^2}_{=1}\gamma^5 = -\operatorname{tr}\gamma^5, \tag{713}$$

  where we used $\left\{\gamma^5, \gamma^0\right\} = 0$ in the second and the cyclicity of the trace in the third step.

- $\operatorname{tr}\gamma^5\gamma^\mu = 0 = \operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\chi$, because $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and because tr(odd #γ's) = 0, as we have already shown.

- $\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu = 0$, because in $4D$ we can always choose and index $\alpha \neq \mu, \nu$ with $\{\gamma^\mu, \gamma^\alpha\} = 0 = \{\gamma^\nu, \gamma^\alpha\}$. As before, $\left\{\gamma^\mu, \gamma^5\right\} = \left\{\gamma^\nu, \gamma^5\right\} = \left\{\gamma^\alpha, \gamma^5\right\} = 0$. Further, we know that

$\gamma^\alpha\gamma_\alpha = \eta^\alpha{}_\alpha = \delta^\alpha{}_\alpha = 4$ in $4D$. Thus

$$\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu = \frac{1}{4}\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu\underbrace{\gamma^\alpha\gamma_\alpha}_{4} \tag{714}$$

$$= \frac{1}{4}(-1)^3\operatorname{tr}\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu\gamma_\alpha \tag{715}$$

$$= -\frac{1}{4}\operatorname{tr}\underbrace{\gamma_\alpha\gamma^\alpha}_{4}\gamma^5\gamma^\mu\gamma^\nu \tag{716}$$

$$= -\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu, \tag{717}$$

where we used the cyclicity of the trace in the third step.

- $\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda = -4i\epsilon^{\mu\nu\kappa\lambda}$, because

$$\operatorname{tr}\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3 = -i\operatorname{tr}\underbrace{\left(\gamma^5\right)^2}_{=\mathbb{1}} = -4i \tag{718}$$

and because the overall result must be totally antisymmetric, when we use that $\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu = 0$.

- $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$ (note that this is a matrix-identity, not a trace identity), because

$$\gamma^\mu\gamma^\nu\gamma_\mu = \underbrace{\{\gamma^\mu,\gamma^\nu\}\gamma_\mu}_{2\eta^{\mu\nu}\gamma_\mu} - \gamma^\nu\underbrace{\gamma^\mu\gamma_\mu}_{4} \tag{719}$$

$$= -2\gamma^\nu. \tag{720}$$

- $\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4\eta^{\rho\sigma}$, because

$$\gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = \underbrace{\{\gamma^\mu,\gamma^\rho\}}_{2\eta^{\mu\rho}}\gamma^\sigma\gamma_\mu - \underbrace{\gamma^\rho\{\gamma^\mu,\gamma^\sigma\}}_{2\eta^{\mu\sigma}}\gamma_\mu + \gamma^\rho\gamma^\sigma\underbrace{\gamma^\mu\gamma_\mu}_{4} \tag{721}$$

$$= 2\gamma^\sigma\gamma^\rho - 2\gamma^\rho\gamma^\sigma + 4\gamma^\rho\gamma^\sigma \tag{722}$$

$$= 2\{\gamma^\sigma,\gamma^\rho\} = 4\eta^{\rho\sigma}. \tag{723}$$

- $\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\kappa\gamma_\mu = -2\gamma^\kappa\gamma^\sigma\gamma^\rho$ (note the reversal in the order of the open indices), because

$$\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\kappa\gamma_\mu = \{\gamma^\mu,\gamma^\rho\}\gamma^\sigma\gamma^\kappa\gamma_\mu - \gamma^\rho\{\gamma^\mu,\gamma^\sigma\}\gamma^\kappa\gamma_\mu + \gamma^\rho\gamma^\sigma\{\gamma^\mu,\gamma^\kappa\}\gamma_\mu - \underbrace{\gamma^\rho\gamma^\sigma\gamma^\kappa\gamma^\mu\gamma_\mu}_{4\gamma^\rho\gamma^\sigma\gamma^\kappa} \tag{724}$$

$$= 2\gamma^\sigma\gamma^\kappa\gamma^\rho - 2\gamma^\rho\gamma^\kappa\gamma^\sigma\underbrace{+2\gamma^\rho\gamma^\sigma\gamma^\kappa - 4\gamma^\rho\gamma^\sigma\gamma^\kappa}_{-2\gamma^\rho\gamma^\sigma\gamma^\kappa} \tag{725}$$

$$= 2\{\gamma^\sigma,\gamma^\kappa\}\gamma^\rho - 2\gamma^\kappa\gamma^\sigma\gamma^\rho - 2\gamma^\rho\{\gamma^\kappa,\gamma^\sigma\} \tag{726}$$

$$= -2\gamma^\kappa\gamma^\sigma\gamma^\rho. \tag{727}$$

Of these many identities, we only need:

$$\operatorname{tr}\big((\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\big) = p'_\kappa p_\lambda\operatorname{tr}\big(\gamma^\kappa\gamma^\mu\gamma^\lambda\gamma^\nu\big) - m_e^2\operatorname{tr}(\gamma^\mu\gamma^\nu) \tag{728}$$

$$= p'_\kappa p_\lambda\big(4\eta^{\kappa\mu}\eta^{\lambda\nu} - 4\eta^{\kappa\lambda}\eta^{\mu\nu} + 4\eta^{\kappa\nu}\eta^{\mu\lambda}\big) - 4m_e^2\eta^{\mu\nu} \tag{729}$$

$$= 4\big(p'^{\mu}p^{\nu} - p'\cdot p\,\eta^{\mu\nu} + p'^{\nu}p^{\mu}\big) - 4m_e^2\eta^{\mu\nu}. \tag{730}$$

Mini-Exercise 33. What is $\mathrm{tr}\Big((\not{k}' + m_\mu)\gamma_\mu(\not{k} - m_\mu)\gamma_\nu\Big)$?

Thus:

$$\begin{aligned}&\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2\\ =&\frac{1}{4}\frac{e^4}{q^4}(4\cdot 4)\cdot\big(p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} - p'\cdot p\,\eta^{\mu\nu} - 4m_e^2\eta^{\mu\nu}\big)\cdot\big(k'_\mu k_\nu + k'_\nu k_\mu - k'\cdot k\,\eta_{\mu\nu} - m_\mu^2\eta_{\mu\nu}\big).\end{aligned} \tag{731}$$

Note: $\frac{m_e^2}{m_\mu^2} \approx \frac{(0.5\text{ MeV})^2}{(100\text{ MeV})^2} \approx \big(\frac{1}{200}\big)^2$, so we can neglect the m_e-term.

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|i\mathcal{M}|^2 = 4\frac{e^4}{q^4}\Big(&\underline{p'\cdot k'\,p\cdot k} + \underline{\underline{p'\cdot k\,p\cdot k'}} - \underbrace{p'\cdot p\,k'\cdot k}\; - 4m_\mu^2 p'\cdot p\\ &+ \underline{\underline{p'\cdot k\,p\cdot k'}} + \underline{p'\cdot k'\,p\cdot k} - \underbrace{p'\cdot p\,k'\cdot k}\; - 4m_\mu^2 p'\cdot p\\ &- \underbrace{p'\cdot p\,k'\cdot k} - \underbrace{p'\cdot p\,k'\cdot k} + \underbrace{4p'\cdot p\,k'\cdot k} + 16m_\mu 2\,p'\cdot p\Big)\end{aligned} \tag{732}$$

$$= 4\frac{e^4}{q^4}\big(2p'\cdot k'\,p\cdot k + 2p'\cdot k\,p\cdot k' + 8m_\mu^2\,p'\cdot p\big). \tag{733}$$

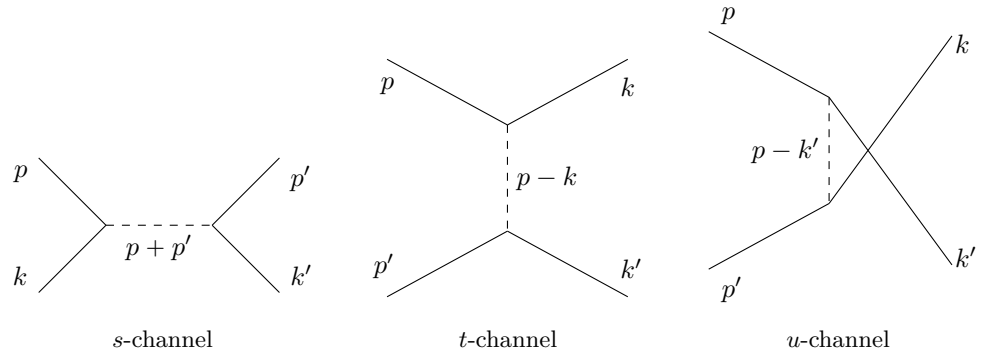
For the rest of the calculation, we actually neglect both masses, i.e., we assume that the center-of-mass energy is much higher then m_μ.
2-2-scattering processes can be expressed in terms of the three *Mandelstam variables*:

$$s = (p'+p)^2 \qquad \text{(which equals the square of the center-of-mass energy)} \tag{734}$$

$$t = (p-k)^2 \qquad \text{and} \tag{735}$$

$$u = (p-k')^2. \tag{736}$$

(For 2-2-scattering of 4 identical particles and a mediator of the interaction coupling through a 3-vertex, there is a diagram for each Mandelstam variable to flow through the internal propagator:



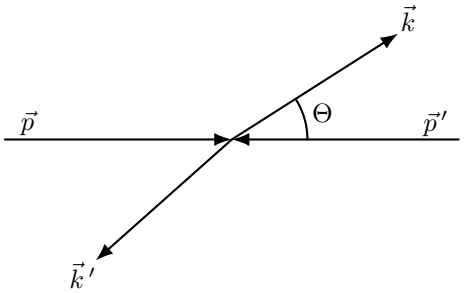
s-channel t-channel u-channel

Because the QED vertex is $\overline{\psi}\gamma^\mu\psi A_\mu$, there is no process with all three diagrams, $e^-e^- \to e^-e^-$ only receives a contribution from t- and u channel.)

For the case $m_e^2 = 0 = m_\mu^2$, which is the approximation that we work with, $p^2 = 0 = p'^2 = k^2 = k'^2$ and thus

$$s = 2p \cdot p' = 2k \cdot k' \tag{737}$$
$$t = -2p \cdot k = -2p' \cdot k' \tag{738}$$
$$u = -2p \cdot k' = -2p' \cdot k. \tag{739}$$

In the center-of-mass frame, these scalar products can all be expressed through the scattering angle Θ,



$$t = -2k \cdot p = -2(k_0 p_0 - \vec{k} \cdot \vec{p}) = -2k_0 p_0 (1 - \cos\Theta) = -\frac{s}{2}(1 - \cos\Theta) \tag{740}$$
$$u = -s - t = -\frac{s}{2}(1 + \cos\Theta) \tag{741}$$

We have that

$$\frac{1}{4}\sum_{\text{spins}} |i\mathcal{M}|^2 = \frac{8e^4}{s^2}\left(\left(\frac{-t}{2}\right)^2 + \left(\frac{-u}{2}\right)^2\right) \tag{742}$$
$$= \frac{2e^4}{s^2}(t^2 + u^2) \tag{743}$$
$$= \frac{2e^4}{s^2}\left(\frac{s^2}{4}(1 - \cos\Theta)^2 + \frac{s^2}{4}(1 + \cos\Theta)\right) \tag{744}$$
$$= \frac{e^4}{2}\big(1 - \cancel{2\cos\Theta} + (\cos\Theta)^2 + 1 + \cancel{2\cos\Theta} + (\cos\Theta)^2\big) \tag{745}$$
$$= e^4\big(1 + (\cos\Theta)^2\big). \tag{746}$$

To arrive at the differential cross-section (see our expression for $d\sigma$ in Eq. (656)), we have to divide by $\frac{1}{4p^0 k^0 |\vec{v}_{\vec{p}} - \vec{v}_{\vec{k}}|} \to \frac{1}{2s}\frac{1}{2c}$ (highly relativistic case, i.e., negligible masses), where we used that $\vec{v}_{\vec{p}} = -\vec{v}_{\vec{k}} \approx c$ in the relativistic case. We will of course continue to work with $c = 1$.

We also have the factor $(2\pi)^4 \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{2k^0} \frac{1}{2k'^0} \delta^4(p + p' - k - k')$, where we are aiming to leave the *angular* distribution non-integrated and thus write $d^3k = d|\vec{k}|\,|\vec{k}|^2\,d\Omega$, with $d\Omega = d\Theta\,d\varphi\,\sin\Theta$ with the scattering angle Θ, which we get by aligning our coordinate system appropriately.
We perform the integration over k' and use $k'^0 = |\vec{k}'|$, which is justified for $m_\mu^2 \ll |\vec{k}'|^2$; we also have $\vec{k}' = -\vec{k}$, which holds in the center-of-mass frame.

Thus

$$(2\pi)^4 \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{2k^0}\frac{1}{2k'^0}\delta^4(\underbrace{p+p'}_{\sqrt{s}}-k-k') = \int \frac{d^3k}{(2\pi)^3}\frac{1}{4|\vec{k}|^2}\delta(\sqrt{s}-2|\vec{k}|) \tag{747}$$

$$= \int \frac{d|\vec{k}|}{(2\pi)^2}\frac{\cancel{|\vec{k}|^2}}{4\cancel{|\vec{k}|^2}}\,d\Omega\,\delta^4(\sqrt{s}-2|\vec{k}|) \tag{748}$$

$$= \frac{1}{32\pi^2}d\Omega \tag{749}$$

$$\implies \frac{d\sigma}{d\Omega} = \frac{1}{4s}\frac{1}{32\pi^2}e^4\big(1+(\cos\Theta)^2\big). \tag{750}$$