Quantum field theory in curved spacetime

Assignment 1 – Apr 28

Exercise 1: Quantum fields in an expanding universe

Motivation: In this first exercise, we'll follow how the vacuum state of a scalar evolves in a toy model of an expanding universe. Even though the setup is simple, it already reveals a key feature of quantum fields in curved spacetime: the vacuum isn't as empty as it seems.

Consider a real massive scalar field χ (minimally coupled) in an expanding universe. Its classical action is

$$S = \frac{1}{2}\int \mathrm{d}^4x \left(\chi'^2 - (\partial_i\chi)^2 - m_{\text{eff}}^2\chi^2\right) , \tag{1.1}$$

where i denotes spatial indices and prime corresponds to derivative with respect to conformal time. The effective mass m_{eff}^2 is written as

$$m_{\text{eff}}^2 = m^2a^2 - \frac{a''}{a} , \tag{1.2}$$

with a, the scale factor. Assume that m_{eff}^2 is given by

$$m_{\text{eff}}^2(\eta) = \begin{cases} m_0^2 , & \eta < 0 \text{ and } \eta > \eta_1 , \\ -m_0^2 , & 0 < \eta < \eta_1 , \end{cases} \tag{1.3}$$

with m_0 a constant.

(**a**) Solve the equations of motion for χ.

(**b**) Construct the early ("in") and late ("out") time vacuum states.

(**c**) Prove that in the "out" region ($\eta > \eta_1$), the state $|0_{\text{in}}\rangle$ (the vacuum in the "in" region) contains particles. In other words, if we initially start in the vacuum the background evolution has created particles.

(**d**) Show that the mean particle number density in a mode $\mathbf{k}$ is given by

$$n_{\mathbf{k}} = \frac{m_0^4}{|k^4 - m_0^4|}\left|\sin\left(\eta_1\sqrt{k^2 - m_0^2}\right)\right|^2 . \tag{1.4}$$

Sanity check: What happens in the limit $\eta_1 \to 0$? Why is this the result we expect?

(**e**) Discuss the regimes $k \gg m_0$ and $k \ll m_0$. What is the physical meaning of these limits?

Exercise 2: Bogolyubov transformations

Motivation: We've seen that one person's vacuum can be filled with particles from another person's point of view. Now we derive general rules that relate the vacua of different observers.

Given a set of mode functions $v_k(\eta)$ (with conformal time η and $k = |\mathbf{k}|$), a scalar field on a cosmological background can be expanded as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}k^3}{(2\pi)^{3/2}} \left(v_k^* a_{\mathbf{k}} e^{i\mathbf{kx}} + v_k a_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right), \tag{2.1}$$

where

$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{2.2}$$

Let us define a new set of mode functions as a linear combination

$$u_k = \alpha_k v_k + \beta_k v_k^*. \tag{2.3}$$

The numbers α_k and β_k are called Bogolyubov coefficients. They are related as

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{2.4}$$

Given the new set of mode functions, we can equivalently expand the scalar as

$$\chi = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}k^3}{(2\pi)^{3/2}} \left(u_k b_{\mathbf{k}} e^{i\mathbf{kx}} + u_k^* b_{\mathbf{k}}^\dagger e^{-i\mathbf{kx}} \right), \tag{2.5}$$

where again

$$[b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta^{(3)}(\mathbf{k}' - \mathbf{k}). \tag{2.6}$$

Then, we can express the different classes of creation and anihilation operators as linear combinations, e.g.

$$b_{\mathbf{k}} = \alpha_k a_{\mathbf{k}} - \beta_k a_{-\mathbf{k}}^\dagger \tag{2.7}$$

In class, you have derived the average particle number density of modes associated with the operator $a^\dagger$ in the b-vacuum. Now, we go a step further and explicitly express the b-vacuum state in terms of creation and anihilation operators of the state a acting on the a-vacuum. The b-vacuum state for a pair of modes $(\mathbf{k}, -\mathbf{k})$ satisfies

$$b_{\mathbf{k}} |0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = b_{-\mathbf{k}} |0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = 0. \tag{2.8}$$

(a) Expand the b-vacuum in terms of a-particle states.

(b) Use the properties of $|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle$ to obtain the expansion coefficients.

(c) Normalize the resulting state. You should obtain the result

$$|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle = \frac{1}{|\alpha_k|} e^{\frac{\beta_k}{\alpha_k} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger} |0_{\mathbf{k},-\mathbf{k}}^{(a)}\rangle. \tag{2.9}$$

(d) Write down the full b-vacuum state in terms of the mode-specific $|0_{\mathbf{k},-\mathbf{k}}^{(b)}\rangle$.

(e) Let's have a closer look at the expansion. What kind of state is the b-vacuum in terms of a-particle states?

Exercise 3: Instantaneous vacuum

Motivation: Every mode function allows to construct a different vacuum. What could be a sensible definition of vacuum then? Let's find out!

Ordinarily, we define the vacuum as the lowest-energy state. In cosmology, however, the Hamiltonian is time dependent. Energy is not conserved. This creates particles. Thus, the lowest-energy state at one time (the *instantaneous vacuum*), may not be the lowest-energy state at a different time. Let's see, how this comes about.

As above consider a real massive scalar field, whose dynamics are characterized by the action given in Eq. (1.1). This results in the Hamiltonian

$$H = \frac{1}{2}\int_x \left(\pi^2 + (\partial_i \chi)^2 + m_{\text{eff}}^2 \chi^2\right), \tag{3.1}$$

with the momentum conjugate π. Assume that the field possesses a mode expansion as in Eq. (2.1).

(a) Express the Hamiltonian in terms of creation and anihilation operators. You should obtain something of the form

$$H = \frac{1}{4}\int \mathrm{d}^3k \left[a_{\mathbf{k}} a_{-\mathbf{k}} F_{\mathbf{k}}^* + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger F_{\mathbf{k}} + (2a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \delta^{(3)}(0))E_{\mathbf{k}}\right] \tag{3.2}$$

for some $E_{\mathbf{k}}$, $F_{\mathbf{k}}$.

(b) Compute the mean energy density in the a-vacuum.

(c) Assuming that $\omega_k^2 = k^2 + m_{\text{eff}}^2 > 0$, find initial conditions for the mode function that minimize the mean energy density at conformal time η_0. (**Hint:** Normalize the mode functions.) What is the corresponding Hamiltonian at conformal time η_0? You should obtain that the Hamiltonian is diagonal in this case.

(d) Compute the initial conditions for the mode function after an infinitesimal time shift, i.e.at conformal time $\eta_0 + \delta\eta$. Compare these initial conditions to the ones derived in the previous exercise. How do we interpret this result? (**Hint:** Have in mind Ex. 2.)

(e) Imagine that you could find a vacuum state which diagonalizes the Hamiltonian at all times. Which equation would the mode functions have to satisfy? Is this equation compatible with the equations of motion?

In specific situations, it can happen that the lowest-energy state at one time η_0 amounts to an infinite number density at a different time η_1, even if the geometry changes slowly compared to the time difference that is characteristic of the problem one would like to answer (i.e.adiabatically). This casts serious doubts on the physical interpretation of the instantaneous vacuum.

However, adiabatic evolution allows us to (at least approximately) define a different vacuum state with interesting properties: The adiabatic vacuum. If the energy density is changing slowly during the considered time interval, the equations of motion allow for the approximate solution[a]

$$v_k^{\text{WKB}}(\eta) = \frac{e^{i\int_{\eta_0}^{\eta} \omega_k(\eta')\mathrm{d}\eta'}}{\sqrt{\omega_k}}. \tag{3.3}$$

We can define the adiabatic vacuum $|0_{\mathrm{ad}}(\eta_0)\rangle$ at a time η_0 by finding exact mode functions which satisfy the initial conditions

$$v_k(\eta_0) = v_k^{\mathrm{WKB}}(\eta_0), \qquad v_k'(\eta_0) = v_k^{\mathrm{WKB}\prime}(\eta_0), \tag{3.4}$$

and constructing the vacua relative to the corresponding anihilation operator.

(f) Quantify how adiabatic a general background evolution yielding $\omega_k(\eta)$ is. Which condition should an adiabatically evolving background satisfy if the quantum-field evolution is considered in a finite-time interval $\Delta\eta = \eta_1 - \eta_0$?

(g) Compute the energy density of the adiabatic vacuum in general. Is it minimal?

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[a] This approximation is called WKB (Wentzel–Kramers–Brillouin) approximation, a standard method in quantum mechanics in general.