Condensed Matter Theory — problem set 7

Dr. Tilman Enss & Valentin Kasper
Institut für Theoretische Physik, Universität Heidelberg
1 June 2016
due date: 10 June 2016
Course homepage: `http://www.thphys.uni-heidelberg.de/~enss/teaching.html`

Problem 18: Hartree-Fock

A uniform spin-s Fermi system has a spin-independent interaction potential $v(\boldsymbol{r}) = (e^2/r)e^{-r/a}$.

(a) Evaluate the self-energy at $T = 0$ in the Hartree-Fock approximation (you may use Mathematica for the last integral). Hence find the excitation spectrum $E_{\boldsymbol{k}}$ and the Fermi energy $\mu = E_{k_F}$.

(b) Show that the exchange contribution to E_{k_F} is negligible for a long-range interaction ($k_F a \gg 1$) but that the direct and exchange terms are comparable for a short-range interaction ($k_F a \ll 1$).

(c) In this approximation prove that the effective mass m^* at the Fermi surface, which is defined by $E_k = E_{k_F} + (k_F/m^*)(k - k_F) + \cdots$, is determined solely by the exchange contribution. Compute m^*, and discuss the limiting cases $k_F a \gg 1$ and $k_F a \ll 1$.

(d) What is the relation between the limit $a \to \infty$ of this model and the electron gas in a uniform positive background?

Problem 19: Potential scattering in 1d

In potential scattering, each particle scatters independently from a fixed potential. The solution can therefore be given exactly, and serves to illustrate how multiple scattering is formulated with geometric series.

(a) Consider a one-dimensional system with single-particle Hamiltonian $\mathcal{H}$ and eigenstates $|\alpha\rangle$,

$$\mathcal{H}|\alpha\rangle = \varepsilon_\alpha|\alpha\rangle\,. \tag{1}$$

The resolvent operator $\mathcal{G}(z)$ is defined by

$$(z\mathbb{1} - \mathcal{H})\,\mathcal{G}(z) = \mathbb{1}\,. \tag{2}$$

Show that $\mathcal{G}(z)$ can be represented in real space as

$$g(x, y; z) = \langle x|\mathcal{G}(z)|y\rangle = \sum_\alpha \frac{\Psi_\alpha(x)\Psi_\alpha^*(y)}{z - \varepsilon_\alpha}\,. \tag{3}$$

[*Note*: The poles of $\mathcal{G}(z)$ as a function of the complex variable z lie on the real axis and determine the eigenvalues of $\mathcal{H}$. $g(x, y; \omega + i0)$ is the retarded Green function.]

(b) Consider specifically a particle which scatters off a δ potential of strength V at position $x = 0$ (in units where $\hbar = 1$),

$$\mathcal{H} = -\frac{\partial_x^2}{2m} + V\delta(x)\,. \tag{4}$$

Show that the (full) Green function can be written as

$$g(x, y; z) = g_0(x, y; z) + g_0(x, 0; z)t(z)g_0(0, y; z) \tag{5}$$

in terms of the noninteracting (free) Green function ($V = 0$)

$$\left(z + \frac{\partial_x^2}{2m}\right)g_0(x, y; z) = \delta(x - y) \tag{6}$$

and the T matrix

$$t(z) = \frac{V}{1 - Vg_0(0, 0; z)}\,. \tag{7}$$

Eq. (5) describes a particle propagating freely between separate scattering events.

(c) Compute the free Green function $g_0(0, 0; z)$ and the T matrix $t(z)$ for $\operatorname{Im} z > 0$. Sketch the frequency dependence of the spectral function $-(1/\pi)\operatorname{Im} t(\omega + i0)$ for positive and negative V. Under which condition does the retarded T matrix $t(\omega + i0)$ have a pole for real ω, and what is the physical meaning of this pole?

[*Hint*: The square root $\sqrt{\omega + i\delta} = i\sqrt{-\omega} + \delta'$ for real $\omega < 0$ and infinitesimal positive $\delta, \delta' > 0$.]

(d*) Compute the *local density of states* at position x and frequency ω,

$$\nu(x, \omega) = -\frac{1}{\pi}\operatorname{Im} g(x, x; \omega + i0)\,. \tag{8}$$

Using your result, show that *(a)* backscattering off the potential leads to spatial oscillations in $\nu(x, \omega)$; and *(b)* for an attractive potential $V < 0$ and for negative frequencies $\omega < 0$ there is a bound state at $x = 0$ which decays exponentially with $|x|$.

[*Intermediate result*: The full Green function reads

$$g(x, x; z) = -\frac{im}{\sqrt{2mz}}\left(1 - \frac{imV}{imV + \sqrt{2mz}}\exp(2i|x|\sqrt{2mz})\right). \tag{9}$$

for $y = x$ and $\operatorname{Im} z > 0$.]

Problem 20: Gaussian integration

Explicitly compute the Gaussian integral for Grassmann variables η_i^*, η_i ($i = 1, 2$) and a 2×2-matrix H_{ij},

$$I = \int d\eta_1^* d\eta_1 d\eta_2^* d\eta_2 \,\exp[-\eta_i^* H_{ij}\eta_j], \tag{10}$$

for instance, by expanding the exponential function into a Taylor series.