EPJ manuscript No.
(will be inserted by the editor)

Shear viscosity and spin sum rules in strongly interacting Fermi gases

Tilman Enss

Physik Department, Technische Universität München, 85747 Garching, Germany

Abstract. Fermi gases with short-range interactions are ubiquitous in ultracold atomic systems. In the absence of spin-flipping processes the number of atoms in each spin species is conserved separately, and we discuss the associated Ward identities. For contact interactions the spin conductivity spectral function $\sigma_{\rm s}(\omega)$ has universal power-law tails at high frequency. We derive the spin f-sum rule and show that it is not affected by these tails in $d < 4$ dimensions. Likewise the shear viscosity spectral function $\eta(\omega)$ has universal tails; in contrast they modify the viscosity sum rule in a characteristic way.

1 Introduction

Condensed matter systems near a phase transition generally have universal low-energy properties, while the high-energy response depends on non-universal details of the microscopic interaction. Ultracold atoms provide an important exception: in dilute gases, where the range of the interaction r_e is much shorter than the mean particle spacing, also the high-energy properties are universal up to a cutoff energy $\hbar^2/mr_e^2$ set by the interaction range, which can be much larger than the Fermi or thermal energies [1]. The correlation functions have characteristic high-frequency and momentum tails which are controlled by the Tan contact density C [2–6]. This quantity measures the probability of finding two atoms of different species near each other. Together, two atoms can absorb a large kinetic energy and undergo a boost in opposite directions while conserving total momentum. Hence, the high-energy response of the system is proportional to the density C of such pairs.

In this work we look in particular at the response to a magnetic field gradient, the spin conductivity $\sigma_{\rm s}$, which has recently been measured [7] and provides an example of quantum limited transport. Aspects of this are understood within kinetic theory [8,9], while a recent strong-coupling Luttinger-Ward calculation [10] explains the spin diffusion quantitatively and predicts the full frequency dependence of the spin conductivity $\sigma_{\rm s}(\omega)$. Furthermore, we consider the response to shear flow, the shear viscosity $\eta(\omega)$ [11–14] in two and three spatial dimensions. The transport coefficients exhibit universal power-law tails at high frequencies, and we study how these tails affect the exact sum rules which link the frequency integrated response to the thermodynamic properties of the system [10,13,15–19]. The question of spin transport is connected with the conservation of the particle numbers N_σ of the spin species. For the case of a density-density interaction there are no spin-flipping processes and each N_σ is conserved separately. This implies spin-selective Ward identities for every spin species, which we will then use to derive the spin sum rule.

2 Model and symmetries

Consider a two-component Fermi gas with contact interaction which is described by the grand canonical Hamiltonian

$$H = \int d\mathbf{x} \sum_\sigma \psi_\sigma^\dagger \Big(-\frac{\nabla^2}{2m} - \mu_\sigma\Big)\psi_\sigma + g_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \tag{1}$$

with mass m, chemical potential μ_σ for spin species $\sigma = \uparrow, \downarrow$, and $\hbar \equiv 1$. At low energies s-wave scattering is allowed only between opposite spins by the Pauli principle. The contact interaction g_0 leads to ultraviolet divergences which need to be regularized [1].

The interacting Fermi gas (1) is invariant under a $U(1) \times U(1)$ symmetry corresponding to the separate conservation of $\uparrow$ and $\downarrow$ particle number. This is readily seen by coupling a different gauge field to each spin component [20]. In the absence of a magnetic field the symmetry is enlarged to $SU(2)$. The spin-selective particle number and current operators can be written as

$$j_\sigma^0(x) = \rho_\sigma(x) = \psi_\sigma^\dagger(x)\psi_\sigma(x) \tag{2}$$

$$j_\sigma^i(x) = -\frac{i}{2m}\Big(\psi_\sigma^\dagger(x)\partial_i\psi_\sigma(x) - \partial_i\psi_\sigma^\dagger(x)\psi_\sigma(x)\Big) \tag{3}$$

where $x = (\mathbf{x}, t)$. These operators satisfy the continuity equation

$$\partial_t \rho_\sigma(x) + \partial_i j_\sigma^i(x) = \partial_\mu j_\sigma^\mu(x) = 0 \tag{4}$$

separately for each spin component with conserved particle numbers $N_\uparrow$ and $N_\downarrow$. The bare current operator (3) acquires no interaction correction in the case of the density-density interaction (1) since $[\rho_\sigma(x), \rho_{\sigma'}(y)] = 0$ and $[H_{\text{int}} - \sum_\sigma \mu_\sigma N_\sigma, \rho_{\sigma'}(x)] = 0$ [21]. The continuity equation implies spin-selective Ward identities which connect the number J_σ^0 and current J_σ^i response functions with the Green's functions. These have been derived by Behn [22],

$$\partial_\mu \langle T j_\sigma^\mu(x)\psi_{\sigma'}(y)\psi_{\sigma'}^\dagger(y')\rangle = \delta_{\sigma\sigma'}\langle T\psi_\sigma(y)\psi_\sigma^\dagger(y')\rangle\,[\delta(x-y) - \delta(x-y')] \tag{5}$$

or in momentum space

$$\begin{aligned}\omega J_\sigma^0(\mathbf{p}, \sigma', \epsilon; \mathbf{p}+\mathbf{q}, \sigma', \epsilon+\omega) - q_i J_\sigma^i(\mathbf{p}, \sigma', \epsilon; \mathbf{p}+\mathbf{q}, \sigma', \epsilon+\omega)& \\ = \delta_{\sigma\sigma'}[G_\sigma(\mathbf{p}+\mathbf{q}, \epsilon+\omega) - G_\sigma(\mathbf{p}, \epsilon)]&\end{aligned} \tag{6}$$

with Green's functions $G_\sigma^{-1}(\mathbf{p}, \omega) = -\omega + \varepsilon_\mathbf{p} - \mu_\sigma - \Sigma_\sigma(\mathbf{p}, \omega)$ and the free single-particle dispersion $\varepsilon_\mathbf{p} = \mathbf{p}^2/2m$. In particular, there is no response of the $\downarrow$ Green's function to a $q_\mu j_\uparrow^\mu$ perturbation, which is not immediately obvious from looking at the perturbative contributions: indeed, Maki-Thompson and Aslamazov-Larkin vertex corrections [13] cancel exactly in this case. For $SU(2)$ invariant models there is an additional Ward identity for the σ^+ operator [22].

3 Spin f-sum rule

The correlation functions of number-current χ_{jn} and spin-current χ_{js} are defined as

$$\chi_{\text{jn/js}}(\mathbf{q}, \omega) = i\int_0^\infty dt \int d\mathbf{x}\, e^{i(\omega^+ t - \mathbf{q}\cdot\mathbf{x})}\langle[j_{\text{n/s}}^z(\mathbf{x}, t), j_{\text{n/s}}^z(\mathbf{0}, 0)]\rangle \tag{7}$$

in terms of the number and spin current operators $j^i_{\mathrm{n/s}}(x) = j^i_\uparrow(x) \pm j^i_\downarrow(x)$ and $\omega^+ = \omega + i0^+$. The corresponding number and spin conductivities in the zero-momentum limit are defined in terms of the retarded correlation function (7) as

$$\sigma_{\mathrm{n/s}}(\omega) = \frac{\mathrm{Im}\,\chi_{\mathrm{jn/js}}(\mathbf{0},\omega)}{\omega}. \tag{8}$$

A Kramers-Kronig transformation relates the frequency integral of $\sigma_{\mathrm{n/s}}(\omega)$ to the current correlation function at zero frequency,

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi} \sigma_{\mathrm{n/s}}(\omega) = \chi_{\mathrm{jn/js}}(\mathbf{0},\omega = 0), \tag{9}$$

which is real. The Kubo formula (7) can be expressed in terms of the fermionic Green's and response functions in the Matsubara formalism as [20,13]

$$\chi_{\mathrm{js}}(\mathbf{0},0) = -\frac{1}{\beta V} \sum_{\mathbf{p}\sigma\sigma'\epsilon_n} \frac{\tau^z_{\sigma\sigma'} p_z}{m} \times \tau^z_{\sigma\sigma'} J^z_\sigma(\mathbf{p},\sigma',i\epsilon_n) = -\frac{1}{\beta V} \sum_{\mathbf{p}\sigma\epsilon_n} \frac{p_z}{m} J^z_\sigma(\mathbf{p},\sigma,i\epsilon_n) \tag{10}$$

where $\tau^z_{\sigma\sigma'} p_z/m$ is the bare spin-current response vertex with Pauli matrix τ^z. This is multiplied with $J^z_\sigma(\mathbf{p},\sigma',i\epsilon_n)$, the fully dressed current response function from Eq. (6) in the limit $\omega = 0, \mathbf{q} \to 0$. The Ward identity (6) for each spin component relates the current response function in the static limit $\omega = 0$, $\mathbf{q} \to 0$ to the Green's function,

$$J^z_\sigma(\mathbf{p},\sigma,i\epsilon_n) = -\frac{\partial G_\sigma(p,i\epsilon_n)}{\partial p_z} = -\frac{p_z}{p}\frac{\partial G_\sigma(p,i\epsilon_n)}{\partial p}. \tag{11}$$

The Matsubara sum over the Green's function yields the momentum distribution $-\beta^{-1}\sum_{\epsilon_n} G(p,i\epsilon_n) = n_{\mathbf{p}\sigma}$ and one obtains

$$\chi_{\mathrm{js}}(\mathbf{0},0) = -\frac{1}{V}\sum_{\mathbf{p}\sigma} \frac{p_z^2}{mp}\frac{\partial n_{p\sigma}}{\partial p} = \chi_{\mathrm{jn}}(\mathbf{0},0). \tag{12}$$

The same result is obtained if one considers not the spin-current but the number-current with bare reponse vertex $\delta_{\sigma\sigma'} p_z/m$. The normalized integral over the solid angle Ω_d yields $\int d\Omega\, p_z^2/\Omega_d = p^2/d$ in $d \geq 2$ dimensions. Integration by parts over p then gives

$$\chi_{\mathrm{jn/js}}(\mathbf{0},0) = -\sum_\sigma \int_0^\Lambda \frac{\Omega_d dp\, p^{d-1}}{(2\pi)^d} \frac{p}{md}\frac{\partial n_{p\sigma}}{\partial p} = \sum_{\mathbf{p}\sigma} \frac{n_{p\sigma}}{m} - \frac{1}{md}\frac{\Omega_d}{(2\pi)^d}\sum_\sigma p^d n_{p\sigma}\Big|_0^\Lambda \tag{13}$$

where we have explicitly written the ultraviolet momentum cutoff $\Lambda \sim 1/|r_e|$. The first term gives the density, while the second term depends on the cutoff. For zero-range interactions the momentum distribution at large momenta is proportional to the Tan contact density, $n_{p\sigma} = C/p^4$ as $p \to \infty$ in any dimension [2–4]. Hence, the cutoff term $C\Lambda^{d-4}$ vanishes for $\Lambda \to \infty$ ($r_e \to 0$) in any dimension $d < 4$. In combination with Eq. (9) this completes the derivation of the particle number and spin sum rule

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega\, \sigma_{\mathrm{n/s}}(\omega) = \frac{n}{m} \tag{14}$$

with the total density $n = n_\uparrow + n_\downarrow$.

In the Galileian invariant model (1) the number current is proportional to momentum and cannot decay. This implies the conservation of the total number current,

$$[H, \int d\mathbf{x}\, j_{\mathrm{n}}^i(x)] = 0, \tag{15}$$

and consequently the number conductivity has a sharp Drude peak at zero frequency,

$$\sigma_{\mathrm{n}}(\omega) = \frac{n}{m}\pi\delta(\omega). \tag{16}$$

In contrast, the global spin current is not conserved because scattering transfers momentum between ↑ and ↓ particles,

$$[H, \int d\mathbf{x}\, j_{\mathrm{s}}^i(x)] \neq 0, \tag{17}$$

and the spin conductivity $\sigma_s(\omega)$ has a finite and nontrivial response at $\omega > 0$. The spin conductivity in 3d has recently been computed in the Luttinger-Ward formalism [10]: there is a broad Drude peak at low frequencies, followed by a universal high-frequency tail

$$\sigma_{\mathrm{s}}(\omega \to \infty) = \frac{C}{3\pi(m\omega)^{3/2}} \qquad (3d) \tag{18}$$

in accordance with results from the operator product expansion [17]. Both in two and three dimensions the tail decays sufficiently fast for the frequency integral (14) to converge, so again the universal high-energy properties of the zero-range model do not affect the form of the spin f-sum rule in $d < 4$ dimensions.

4 Shear viscosity sum rule

The shear viscosity η measures the friction of a fluid subject to a shear flow of both spin species simultaneously (mass flow). The real part of the frequency-dependent shear viscosity,

$$\eta(\omega) = \frac{\operatorname{Im}\chi_{xyxy}(\omega)}{\omega} \tag{19}$$

is defined via the retarded stress correlation function

$$\chi_{xyxy}(\mathbf{q},\omega) = i\int_0^\infty dt \int d\mathbf{x}\, e^{i(\omega^+ t - \mathbf{q}\cdot\mathbf{x})}\langle[\Pi_{xy}(\mathbf{x},t), \Pi_{xy}(\mathbf{0},0)]\rangle \tag{20}$$

at zero external momentum, $\mathbf{q} = 0$. In general the real shear viscosity contains an additional contact term proportional to $\delta(\omega)$ [23], however in our case of an interacting Fermi gas at $T > 0$ this is canceled by the real part of $\chi_{xyxy}(\omega = 0)$ and does not appear explicitly. The stress tensor operator has the off-diagonal components [13,21]

$$\Pi_{xy} = \sum_{\mathbf{p}} \frac{p_x p_y}{m} c^\dagger_{\mathbf{p}-\mathbf{q}/2,\sigma} c_{\mathbf{p}+\mathbf{q}/2,\sigma} \tag{21}$$

since the interaction correction vanishes in the zero-range limit [13,21]. Again a Kramers-Kronig transformation relates the frequency integral of the viscosity to the stress correlation function at zero external frequency (static limit),

$$\int_{-\infty}^{\infty} \frac{d\omega}{\pi}\eta(\omega) = \chi_{xyxy}(\omega = 0). \tag{22}$$

In analogy with the spin case, the Kubo formula (20) is expressed in terms of the stress response function T_{xy} as [13]

$$\chi_{xyxy}(0) = -\frac{1}{\beta V} \sum_{\mathbf{p}\sigma\epsilon_n} \frac{p_x p_y}{m} T_{xy}(\mathbf{p}, i\epsilon_n)\,. \tag{23}$$

In the static limit of external $\omega = 0, \mathbf{q} \to 0$ the stress response is determined by the Ward identity associated with momentum current conservation [24],

$$T_{xy}(\mathbf{p}, i\epsilon_n) = -p_x \frac{\partial G(p, i\epsilon_n)}{\partial p_y} = -\frac{p_x p_y}{p} \frac{\partial G(p, i\epsilon_n)}{\partial p} \tag{24}$$

and hence

$$\chi_{xyxy}(0) = -\frac{1}{V} \sum_{\mathbf{p}\sigma} \frac{p_x^2 p_y^2}{mp} \frac{\partial n_{p\sigma}}{\partial p}. \tag{25}$$

The normalized integral over the solid angle Ω_d yields $\int d\Omega_d\, p_x^2 p_y^2/\Omega_d = p^4/[d(d+2)]$ in $d \geq 2$ dimensions. Performing an integration by parts as in Eq. (13) relates the correlation function to the kinetic energy density $E_{\text{kin}} = \frac{1}{V}\sum_{\mathbf{p}\sigma} \varepsilon_p n_{p\sigma}$. The integrals are ultraviolet divergent and can be regularized by a momentum cutoff Λ,

$$\chi_{xyxy}(0) = \frac{2}{d} E_{\text{kin}} - \frac{1}{md(d+2)} \frac{\Omega_d}{(2\pi)^d} \sum_\sigma p^{d+2} n_{p\sigma}\Big|_0^\Lambda. \tag{26}$$

Through the momentum distribution $n_{p\sigma} = C/p^4$ for $p \to \infty$ (see above) the cutoff term depends on the contact density C,

$$\chi_{xyxy}(0) = \frac{2}{d} E_{\text{kin}} - \frac{\Omega_d}{(2\pi)^d} \frac{C\Lambda^{d-2}}{md(d+2)}. \tag{27}$$

The kinetic energy density E_{kin} can be re-written using the Tan relations for the internal energy density ε or the pressure P as [2,3]

$$E_{\text{kin}} = \varepsilon + \frac{C}{4\pi m} \ln\frac{\omega_\Lambda}{\varepsilon_B} = P - \frac{C}{4\pi m} + \frac{C}{4\pi m} \ln\frac{\omega_\Lambda}{\varepsilon_B} \qquad (2d) \tag{28}$$

$$E_{\text{kin}} = \varepsilon - \frac{C}{4\pi m}\left(\frac{1}{a} - \frac{2\Lambda}{\pi}\right) = \frac{3}{2}\left[P - \frac{C}{4\pi ma} + \frac{C\Lambda}{3\pi^2 m}\right] \qquad (3d) \tag{29}$$

with the cutoff energy $\omega_\Lambda = 2\varepsilon_\Lambda = \Lambda^2/m$ and the two-particle binding energy ε_B. Then the stress correlation function including the cutoff term in Eq. (27) reads

$$\chi_{xyxy}(0) = P - \frac{3C}{8\pi m} + \frac{C}{4\pi m} \ln\frac{\omega_\Lambda}{\varepsilon_B} \qquad (2d) \tag{30}$$

$$\chi_{xyxy}(0) = P - \frac{C}{4\pi ma} + \frac{4C\sqrt{m\omega_\Lambda}}{15\pi^2 m} \qquad (3d). \tag{31}$$

The zero-range interaction leads to universal high-frequency tails $\eta(\omega) \sim C/(8m\omega)$ in 2d [17,18] and $\eta(\omega) \sim C/(15\pi\sqrt{m\omega})$ in 3d [13,16]. These tails have to be subtracted to make the frequency integral (22) convergent, and one obtains the shear viscosity sum rules [13,16,19]

$$\frac{2}{\pi}\int_0^\infty d\omega \left[\eta(\omega) - \frac{C}{8m\omega}\Theta(\omega - \varepsilon_B)\right] = P - \frac{3C}{8\pi m} = \varepsilon - \frac{C}{8\pi m} \qquad (2d) \tag{32}$$

$$\frac{2}{\pi}\int_0^\infty d\omega \left[\eta(\omega) - \frac{C}{15\pi\sqrt{m\omega}}\right] = P - \frac{C}{4\pi ma} = \frac{2}{3}\varepsilon - \frac{C}{6\pi ma} \qquad (3d). \tag{33}$$

The universal high-frequency behavior is most clearly seen if one looks near the quantum critical point at zero density and zero temperature [25, 26]. The shear viscosity in this limit but with the same value of C as in the dense system has the form [19]

$$\eta_0(\omega) = \frac{C}{8m\omega}\left(1 - \frac{\varepsilon_B}{\omega}\right)^2 \Theta(\omega - \varepsilon_B) \qquad (2d). \tag{34}$$

By subtracting $\eta_0(\omega)$ one arrives a low-energy sum rule which captures only the finite-density effects [19]

$$\frac{2}{\pi}\int_0^\infty d\omega\,[\eta(\omega) - \eta_0(\omega)] = P \qquad (2d). \tag{35}$$

In conclusion, we have argued that zero-range interactions realized in ultracold atomic systems do not modify the spin f-sum rule but lead to characteristic contact terms in the shear viscosity.

References

1. I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. **80**, 885 (2008).
2. S. Tan, Ann. Phys. (NY) **323**, 2971 (2008); S. Tan, Ann. Phys. (NY) **323**, 2952 (2008).
3. F. Werner and Y. Castin, arXiv:1001.0774; F. Werner and Y. Castin, Phys. Rev. A **86**, 013626 (2012).
4. M. Barth and W. Zwerger, Ann. Phys. (NY) **326**, 2544 (2011).
5. M. Valiente, N. T. Zinner, and K. Mølmer, Phys. Rev. A **84**, 063626 (2011).
6. E. Braaten, in *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger (Springer, 2012), p. 193.
7. A. Sommer, M. Ku, G. Roati, and M. W. Zwierlein, Nature (London) **472**, 201 (2011).
8. G. M. Bruun, Phys. Rev. A **85**, 013636 (2012).
9. T. Enss, C. Küppersbusch, and L. Fritz, Phys. Rev. A **86**, 013617 (2012).
10. T. Enss and R. Haussmann, Phys. Rev. Lett. **109**, 195303 (2012).
11. C. Cao, E. Elliott, J. Joseph, H. Wu, J. Petricka, T. Schäfer, and J. E. Thomas, Science **331**, 58 (2011).
12. P. Massignan, G. M. Bruun, and H. Smith, Phys. Rev. A **71**, 033607 (2005).
13. T. Enss, R. Haussmann, and W. Zwerger, Ann. Phys. (NY) **326**, 770 (2011).
14. C. Chafin and T. Schäfer, arXiv:1209.1006; P. Romatschke and R. E. Young, arXiv:1209.1604.
15. L. P. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Oxford University Press, 2003).
16. E. Taylor and M. Randeria, Phys. Rev. A **81**, 053610 (2010).
17. J. Hofmann, Phys. Rev. A **84**, 043603 (2011).
18. W. D. Goldberger and Z. U. Khandker, Phys. Rev. A **85**, 013624 (2012).
19. E. Taylor and M. Randeria, Phys. Rev. Lett. **109**, 135301 (2012).
20. A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of quantum field theory in statistical physics* (Dover, 1975).
21. Y. Nishida and D. T. Son, Phys. Rev. D **76**, 086004 (2007).
22. U. Behn, physica status solidi (b) **88**, 699 (1978).
23. B. Bradlyn, M. Goldstein, and N. Read, Phys. Rev. B **86**, 245309 (2012).
24. A. M. Polyakov, JETP **30**, 1164 (1969) [Zh. Eksp. Teor. Fiz. **57**, 2144 (1969)].
25. P. Nikolić and S. Sachdev, Phys. Rev. A **75**, 033608 (2007).
26. T. Enss, Phys. Rev. A **86**, 013616 (2012).