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## 1 Introduction

A time-dependent analytical description of particle production from the available relativistic energy in heavy-ion collisions is of considerable interest. In particular, the accurate modeling of transverse momentum and rapidity distribution functions for produced particles is a basic requirement in attempts to understand the relevant partonic and hadronic physical processes. Analytically solvable models offer transparent approaches to the problem, including the possibility to extrapolate to other incident energies, such as from RHIC energies  $\sqrt{s_{NN}} = 19.6$ –200 GeV to LHC,  $\sqrt{s_{NN}} = 5.52$  TeV.

In this work I propose a nonequilibrium-statistical (and hence, time-dependent) approach that provides analytical solutions for pseudorapidity distributions of produced particles. It is based on a nonlinear diffusion equation in rapidity space, which accounts for the explicit dependence of the diffusion coefficient on the rapidity density in the initial short ( $\tau_p \simeq 0.25$  fm/c) partonic phase of the collision when most of the particles are produced. This is followed by the somewhat more extended phase of color neutralization [16], and a long-lasting color-neutral pre-hadronic or hadronic phase with rapid expansion of the system.

Three sources for particle production are considered, two for initial rapidities close to the beam values, and a third central source that arises mostly from gluon-gluon collisions. I consider the transition from the initial, highly nonlinear partonic phase to the subsequent, essentially linear phase. Other investigations such as [17], and references to numerical simulations therein, corroborate the short duration of the parton-production phase, and the long duration of the subsequent recombination (pre-hadronic) and hadronic phase. According to parton-cascade models such as [18], the pre-hadrons are color singlets that are generated from quark and gluon recombination in a statistical coalescence process. They decay into the final hadrons according to their relative phase-space weights. Many numerical approaches to the problem

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use string models in the initial phase, and final-state hadronic collisions, but often hadronic rescattering is not considered.

The second phase lasts about  $\tau_{\text{int}} \simeq 7\text{--}10$  fm/c (depending on the system, the incident energy, and the centrality). Phenomenologically, it is well accounted for in a Relativistic Diffusion Model (RDM) based on a linear Fokker-Planck equation with constant diffusion coefficient, plus fast collective expansion. The RDM had been developed some time ago [1–5] and compared in detail with data on net proton rapidity distributions at SIS, AGS, SPS, and RHIC energies, and with produced-particle distributions. For pseudo-rapidity distributions of produced charged hadrons, a  $\chi^2$ -optimization of the Jacobi-transformed analytical solutions yields very precise agreement with the available RHIC data provided the midrapidity source for particle production is taken into account.

For net-proton rapidity distributions at RHIC energies, there have been several investigations of non-linear effects within the diffusion approach. A nonlinearity in the drift coefficient that secures the correct Maxwell-Boltzmann equilibrium limit for  $t \rightarrow \infty$  has been investigated in [8], but the deviations from linearity are small.

The strong effect of a nonlinearity in the diffusion coefficient that also persists over the full interaction time of typically 10 fm/c [14, 15] as a consequence of the introduction of non-extensive statistics [10] had been investigated in [8, 39] for net protons in heavy-ion collisions, and in [9] for produced particles in  $p\bar{p}$ -collisions. It seems that this is a way to account for the collective expansion of the system without considering it explicitly. If one includes an explicit treatment of collective expansion [40], however, there is no need to introduce non-extensive statistics when comparing the RDM-results with data, the linear evolution after the parton-production phase yields excellent results for both net baryons, and produced particles.

The linear diffusion model had also been proposed in order to calculate and predict transverse energy distributions of hadrons [6], and more recently to calculate transverse momentum distributions of identified hadrons (neutral pions, as well as negative pions and kaons) in a nonequilibrium-statistical approach including radial flow [7].

Equilibrium-statistical models [11–13] account in remarkable detail for relative production rates of produced particles at central rapidity with only the temperature and the chemical potential as parameters. Due to the lack of time dependence - and consequently, of nonequilibrium-statistical effects - a relevant ingredient is, however, missing if one aims at the precise modeling of distribution functions.

The gradual approach of the system towards statistical equilibrium in the course of a relativistic heavy-ion collisions is presented in a schematic analytical model in this work. I start with the nonlinear diffusion model as expected to be valid during the parton-production phase in Sect. 2, followed by a consideration of the so-called source solution of the nonlinear problem in Sect. 3. For sufficiently large times, the initial power-law behaviour during the nonlinear parton-production phase is superseded by the essentially linear diffusion (RDM-) phase in rapidity space which produces gaussian tails in pseudorapidity space, Sect. 4.

The late-stage time evolution is discussed in a comparison with RHIC data for the asymmetric d + Au-system, which indeed show the gaussian tails. In an asymmetric system like d+Au, the nonequilibrium effects are visible more directly than in case of symmetric systems such as Au + Au. The interaction ceases long before statistical equilibrium with respect to the variable rapidity is reached. The conclusions are drawn in Sect. 5.

## 2 The nonlinear diffusion equation

The origin of diffusion during and after particle production in a heavy-ion reaction at relativistic energies is found in momentum space, through random momentum kicks of the produced particles – partons in the soft-gluon field in the early stage, prehadrons and hadrons in later stages. Diffusion in coordinate space appears as a secondary effect.

The corresponding fluctuations can be seen in rapidity and pseudorapidity distributions of produced particles [1–5], as well as in transverse energy and momentum distributions [6, 7]. To provide an analytical treatment of the problem in both early and late stages, I confine the present work to rapidity space, with the lorentz-invariant rapidity  $y = 0.5 \cdot \ln((E + p)/(E - p))$ , and a subsequent Jacobian transformation to pseudorapidity  $\eta$  that is required for the comparison to the available data for produced charged hadrons.

To incorporate the early parton-production phase into the relativistic diffusion model [1–3], a dependence of the diffusion coefficient on the initially very high rapidity density  $R(y, t)$  has to be considered,  $D_y \rightarrow D_y(R)$ , such that the linear transport equation in rapidity space that I investigated in [1] is replaced by

$$\frac{\partial}{\partial t} R(y, t) = -\nabla_y \left[ J(y) R(y, t) \right] + \nabla_y D_y(R(y, t)) \nabla_y R(y, t). \quad (1)$$

The drift term  $J(y)$  governs the gradual approach of the mean values towards statistical equilibrium. The diffusion coefficient  $D_y(R)$  depends on the rapidity density and hence, the equation is generally highly nonlinear. It is therefore expected to account not only for the long-lasting, essentially linear diffusive phase as in [1–3], but also for the partonic initial phase of high rapidity density. In this short-lived phase the major part of particle production with rapidly rising norm of the distribution function takes place. The rising norm is phenomenologically accounted for in this work by letting the integration constant in Eq.(1) depend on particle number.

In case of the linear RDM [1, 6] with  $D_y(R) = D_y = \text{const.}$ , I had assumed an instant production of the particles in the three sources, and subsequent diffusion in  $y$ -space during the interaction time. This initial condition is exactly fulfilled only for net baryons [1]. However, for produced charged hadrons, it also yields extremely precise results when compared [2, 3, 5] in detail to the data. An explicit treatment of the short nonlinear parton production phase with a strong dependence of the diffusion coefficient on the distribution function should therefore preserve the model features of the subsequent, essentially linear diffusive phase.

To account for the strong correlation between diffusion coefficient and rapidity density distribution in the initial high-density particle production phase, I propose a dependence on a power  $\kappa$  of the rapidity density according to

$$D_y[R(y, t)] = D_y^p \cdot R(y, t)^\kappa \quad (2)$$

with  $D_y^p$  the rapidity diffusion constant in the particle production phase. For certain critical exponents  $\kappa$ , analytical solutions of the diffusive part of the transport equation can be obtained.

In a two-step approach, the subsequent – probably mostly pre-hadronic and hadronic – evolution in pseudorapidity space during the interaction time of  $\tau_{\text{int}} \simeq 7\text{--}10$  fm/c (mean duration of the collision; see [14]) is accounted for in the Relativistic Diffusion Model [1] (RDM,  $\kappa = 0$ ) with a linear drift term  $J(y) = (y_{\text{eq}} - y)/\tau_y$  governed by the rapidity relaxation time  $\tau_y$  and the equilibrium value of the rapidity  $y_{\text{eq}}$ . The diffusion term is here  $\propto D_y \frac{\partial^2}{\partial y^2} R(y, t)$ .

The diffusion constant  $D_y$  in this phase is significantly smaller than the value of  $D_y^p$  in the short production phase with a large number of (partonic) degrees of freedom. The linear model with instant particle production yields excellent agreement with d+Au, Cu+Cu and Au+Au data at RHIC energies, including the detailed centrality dependence [3].

The initial short, highly nonlinear phase of parton production occurs within  $\tau_p \simeq 0.25$  fm/c [17] in two beam-like sources, and a central source in rapidity space. Since the time scale for particle production in all three sources is faster than the one for the nonlinear diffusion, it turns out that the particle content in the power-law tails remains small: there is little spread of the distribution function in rapidity space in this initial phase.

The mathematical treatment of the initial nonlinear phase is confined to the central source as an example. The beam-like sources are then dealt with in an analogous way, but with the freedom to choose different diffusion coefficients because the production mechanisms in the valence-quark dominated beam-like regions of rapidity space are different from the central region with few valence quarks at RHIC or LHC energies. Kinematic constraints in the beam-like sources expected at high absolute values of rapidity are not considered here.

During the short production phase, the drift in  $y$ -space is not yet pronounced, and I therefore treat here only the diffusive part of the nonlinear transport equation,  $R(y, t) \rightarrow P(y, t)$  with

$$\frac{\partial}{\partial t} P(y, t) = D_y^p \nabla_y P(y, t)^\kappa \nabla_y P(y, t). \quad (3)$$

The solution of this equation at the end of the nonlinear production phase ( $t = \tau_p$ ) can then be used as initial condition for the subsequent linear diffusive time evolution treated in the Relativistic Diffusion Model (RDM) [1–5],

$$R_0(y, 0) = P(y, \tau_p). \quad (4)$$

With  $t^* = t \cdot D_y^p$  the nonlinear diffusion equation becomes

$$\frac{\partial}{\partial t^*} P(y, t^*) = \nabla_y P(y, t^*)^\kappa \nabla_y P(y, t^*). \quad (5)$$

This equation has been extensively studied in many diverse areas of science, and a large amount of mathematical literature exists, [19–27], and references therein. It has mostly been considered for positive values of  $\kappa$  such as  $\kappa = 1$  for thin saturated regions in porous media,  $\kappa \geq 1$  for the percolation of gas through porous media,  $\kappa = 3$  for thin films spreading under gravity, and  $\kappa = 6$  for radiative heat transfer by Marshak waves. Many problems are dealt with in only one (spatial) dimension, analogous to the present work which is confined to one (momentum-like) dimension. It has also been established that the number of exact solutions is limited. Solutions for several power-law diffusivities with negative  $\kappa$  are known [25], in particular, for  $\kappa = -1/2, -1, -4/3, -3/2$  and  $-2$ .

The physically most interesting behaviour occurs in the region of small diffusion coefficients  $D(R)$ , where a moving boundary may exist. The behaviour of solutions for positive and negative values of  $\kappa$  is distinctively different. Depending on the specific values for the constants of integration (see below), a free boundary may occur for  $\kappa = 1$  that has a finite gradient and moves with finite velocity, similarly for other positive values of  $\kappa$ , but for  $\kappa > 1$ , the gradient at the boundary becomes infinite.

For  $\kappa = -1$  and some other negative values, however, there is an instantaneous spread without a free boundary, as was treated by Pattle [19], Pert [22] Hill [25] and others for instantaneous heat deposition in a medium with concentration-dependent diffusion coefficient. This situation is in some respect analogous to the initial parton production from the available energy in a relativistic heavy-ion collision. Here the initial rapidity density distribution of the created partons should not have a free boundary in rapidity space, except for kinematical constraints.

Hence I investigate solutions for negative values of  $\kappa$  emphasizing  $\kappa = -1$ . An exact solution is not only useful to test the accuracy of numerical results, but it is also important to understand and describe the physical behaviour of the system. In particular, the analytical solution at the end of the initial nonlinear phase may then be used as initial condition for the subsequent, essentially linear diffusion process in  $y$ -space which can be modeled analytically within the given simple but successful RDM-framework.

The majority of known exact solutions of the nonlinear problem are so-called similarity solutions [19–26]: With an assumed functional form of the solution, the partial differential equation reduces to an ordinary differential equation, or to a partial differential equation of lower order, which can then be integrated in closed form under certain conditions. The similarity solution of (5) for  $t^* \rightarrow t$  is written as

$$P(y, t) = y^{2\lambda/\kappa(1+\lambda)} \Phi(\xi) \quad (6)$$

with

$$\xi = y^{1/(1+\lambda)}/t^{1/2} \quad (7)$$

and  $\lambda$  is an arbitrary constant with  $\lambda \neq -1$ . Inserting this ansatz into the nonlinear differential equation (5) yields first integrals for two values of  $\lambda$  (for  $\kappa = -1$  and  $-2$ , only one value of  $\lambda$ )

$$\lambda_1 = -\kappa/(\kappa + 2) \quad (8)$$

$$\lambda_2 = -\kappa/(\kappa + 1) \quad (9)$$

which have been given by Hill and Hill (1990) [25] as

$$\frac{\Phi^\kappa \Phi'}{\xi} - \frac{2\Phi^{\kappa+1}}{(\kappa + 2)\xi^2} + \frac{2\Phi}{(\kappa + 2)^2} = C_1 \quad (10)$$

$$\frac{\Phi^\kappa \Phi'}{\xi} - \frac{(2\kappa + 3)\Phi^{\kappa+1}}{(\kappa + 1)^2\xi^2} + \frac{\Phi}{(2\kappa + 1)^2} = C_1 \quad (11)$$

with  $\Phi' = \partial\Phi/\partial\xi$ . For vanishing constants of integration  $C_1 = 0$ , the so-called "source" and "dipole" solutions arise from these first integrals. For finite  $C_1 \neq 0$ , a number of exact solutions for special values of  $\kappa$  have also been derived [25]. Here I investigate the source solution for  $C_1 = 0$

$$\Phi = \left( C_2 \xi^{2\kappa/(\kappa+2)} - \frac{\kappa \xi^2}{2(\kappa + 2)} \right)^{1/\kappa}. \quad (12)$$

This solution is not defined for  $\kappa = -2$ , and becomes singular for  $\kappa < -2$ . Hence, a solution without a free boundary as required for the initial rapidity diffusion problem can in principle only occur for  $-2 < \kappa < 0$ . The desired solution for  $\kappa = -1$  (and hence,  $\lambda = \lambda_1 = 1$ ) obeys the first-order partial differential equation

$$\frac{\Phi'}{\xi\Phi} - \frac{2}{\xi^2} + 2\Phi = 0 \quad (13)$$

and the solution of the nonlinear diffusion equation

$$P(y, t) = \Phi(\xi)/y \quad (14)$$

with  $\xi = (y/t)^{1/2}$  becomes

$$P(y, t) = [C_2 y \xi^{-1} + y \xi^2/2]^{-1} \quad (15)$$

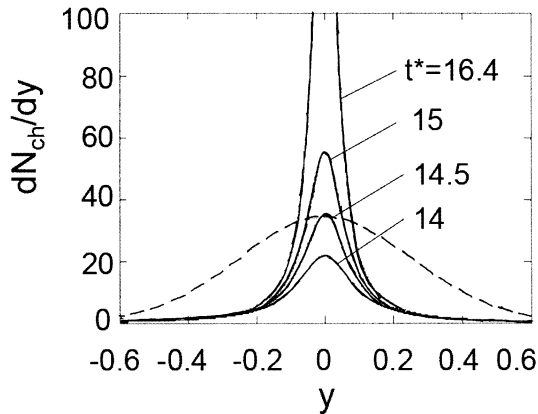
where  $C_2$  is the constant of integration.

### 3 The source solution in particle production

The solution of the nonlinear diffusion problem in a high-density phase such as during parton production in rapidity space has thus been reduced to

$$P(y, t) = [C_2 t + y^2/(2t)]^{-1}. \quad (16)$$

To account for the increasing norm of the rapidity distribution function during the rapid parton production process, I let the integration constant  $C_2$  depend on the particle number. Since the particle number is likely to increase exponentially with time during the first collision phase where  $t \equiv \hat{t} < \tau_p \simeq 0.25$  fm/c, I take



**Fig. 1** Time evolution of rapidity distributions for produced charged particles from minimum-bias d + Au collisions at  $\sqrt{s_{NN}} = 200$  GeV in the initial nonlinear phase for the midrapidity source. The norm of the distribution is rising with increasing  $t^* = D_y^p \cdot t$  until at  $t = \tau_p$  all the particles have been produced and the linear diffusion process in  $y$ -space starts. The dashed curve is a Gaussian with the same particle content  $N_{ch}^3 = 22$  as the power-law solution of the nonlinear problem at  $t = \tau_p$ .

the particle-number dependence into account phenomenologically by choosing the integration constant in the denominator as

$$C_2 = \exp(-t). \quad (17)$$

In the present two-step model, all particles are assumed to be produced until  $t = \tau_p$  where the nonlinear production phase turns into an (essentially linear) diffusion process with  $\kappa = 0$ . That is, for  $C_2 = \exp(-\tau_p)$  the norm of the solution reaches its maximum value

$$\int P(y, t = \tau_p) dy = 1. \quad (18)$$

The corresponding solutions (16) of the nonlinear diffusion problem are shown in Fig. 1 for the central source of parton production in d+Au at  $\sqrt{s_{NN}} = 200$  GeV. Results of the rapidity distribution functions for produced particles multiplied with the number of charged hadrons produced in the central source ( $N_{ch}^3 = 22$ , [3]) are shown for four values of  $t^* = t \cdot D_y^p = 14, 14.5, 15, 16.4$ , with the norm of  $P(y, t)$  reaching  $\int P(y, t) dy = 1$  for  $t = \tau_p = 0.25$  fm/c at  $t^* = 16.4$ .

The distribution functions are seen to rise strongly with increasing  $t^*$  in a narrow midrapidity region due to the fast increase of the particle number, with an only moderate increase in the power-law tails where the particles with higher rapidities are created: these tails are already present at very short times. Physically, the partons with the highest rapidity values are created already at the shortest times.

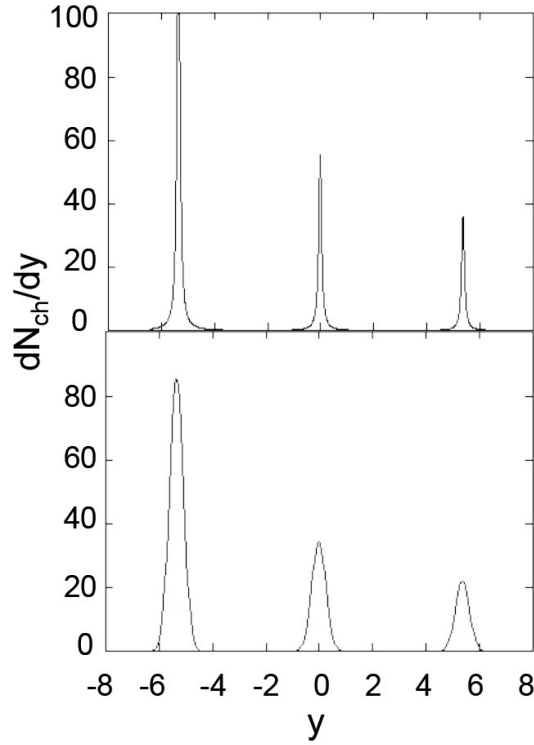
The rapidity diffusion coefficient in the particle production phase is thus  $D_y^p = 16.4/\tau_p \simeq (16.4/0.25)c/\text{fm} \simeq 66c/\text{fm}$ , which is significantly larger than the diffusion coefficient  $D_y$  in the subsequent long-lasting ( $\simeq 7$  fm/c) linear diffusion phase. This reflects the larger number of degrees of freedom in the initial phase which is mainly partonic, and the higher density of particles.

The dashed curve in Fig. 1 is a Gaussian that arises from a linear time evolution ( $\kappa = 0$ ) and  $\delta$ -function initial conditions with instant particle production at  $t = 0$ . It has the same particle content (22 charged hadrons) as the nonlinear solution at  $t^* = \tau_p D_y^p = 16.4$ .

For the beam-like sources at initial rapidities  $y_{1,2} = \mp y_{\max}$ , the solutions of the nonlinear problem for  $\kappa = -1$  are accordingly ( $t^* = t \cdot D_{y1,2}^p \rightarrow t$ )

$$P_{1,2}(y, t) = [C_2 t + (y \pm y_{\max})^2 / (2t)]^{-1}. \quad (19)$$

Here the diffusion coefficients in the particle production phase  $D_{y1,2}^p$  are likely to differ from  $D_y^p$  in the midrapidity region because the microscopic processes during particle production are substantially different, with the diffusion coefficient at midrapidity mostly due to gluon-gluon collisions. The analytical solutions at the end of the initial phase  $t = \tau_p$  in the three sources are displayed in the upper part of Fig. 2 for minimum-bias d+Au at  $\sqrt{s_{NN}} = 200$  GeV.



**Fig. 2** Rapidity distributions for produced charged particles from minimum-bias d + Au collisions at  $\sqrt{s_{NN}} = 200$  GeV at the end of the initial nonlinear phase (top) for the three sources, with particle contents  $N_{\text{ch}}^1 = 55$ ,  $N_{\text{ch}}^2 = 14$ , and  $N_{\text{ch}}^3 = 22$ . In the bottom frame, corresponding Gaussians are displayed (see also Fig. 3). They arise from a linear evolution with  $\delta$ -function initial conditions.

The corresponding number of charged hadrons [3] created in the Au-like source is  $N_{\text{ch}}^1 = 55$ , in the  $d$ -like source  $N_{\text{ch}}^2 = 14$ , and in the central source  $N_{\text{ch}}^3 = 22$ , with a total of 91 produced charged hadrons in minimum-bias collisions. These particle numbers have been determined from a detailed comparison of the subsequent linear diffusion phase with data, see Sect. 4. In the lower part of Fig. 2, Gaussians with the same particle-number content are displayed, as they arise from a linear diffusive time evolution ( $\kappa = 0$ ) with  $\delta$ -function initial conditions, see next section.

At  $t = \tau_p$  the power-law solution of the initial nonlinear phase in the midrapidity source can be expressed as

$$P_3(y, t) = \frac{C}{a^2 + y^2} \quad (20)$$

with

$$C = 2\tau_p D_y^p \quad (21)$$

$$a = \tau_p D_y^p \sqrt{2 \exp(-\tau_p D_y^p)} \quad (22)$$

for the central source, and analogously  $P_{1,2}(y, t)$  for the beam-like sources with  $y \rightarrow y \pm y_{\text{max}}$  and  $D_y^p \rightarrow D_{y1,2}^p$ .

As was shown in [1–5], the linear diffusive evolution after the initial short parton production phase is in very good agreement with the available data. Hence the power-law result (20) of the first phase can be used as an initial condition for the second, essentially linear phase that is reconsidered as described by Eq. (31) in the next section. With this initial condition, the solution of the linear diffusion problem for the central source becomes [28]

$$R(y, t) = \frac{C}{2a} \sqrt{\frac{2\pi}{\sigma_y^2}} \Re \left[ \exp \left[ -\frac{(y + iv)^2}{2\sigma_y^2} \right] \operatorname{erfc} \left( \frac{iy - v}{\sqrt{2\sigma_y^2}} \right) \right] \quad (23)$$



with the variance

$$\sigma_y^2(t) = D_y^p \tau_y [1 - \exp(-2t/\tau_y)] \quad (24)$$

and

$$v = a \exp(-t/\tau_y). \quad (25)$$

For  $t \rightarrow \infty$  we have

$$\Re \left[ \operatorname{erfc} \left( \frac{iy}{\sqrt{2\sigma_y^2}} \right) \right] = 1 \quad (26)$$

such that the gaussian limit is attained

$$R(y, t \rightarrow \infty) = \sqrt{\frac{1}{2\pi\sigma_y^2}} \exp \left[ -\frac{y^2}{2\sigma_y^2} \right] \quad (27)$$

which is a solution of a linear Fokker-Planck equation (31) for  $t \rightarrow \infty$  with  $\delta$ -function initial conditions. For gaussian initial conditions with an initial variance  $\sigma_0^2$ , one obtains a gaussian solution with  $\sigma_y^2(t) \rightarrow \sigma_0^2 + \sigma_0^2 \exp(-2t/\tau_y)$  and hence, Eq.(27) results for  $t \rightarrow \infty$  as well.

To connect the diffusion approach with data, the linear Relativistic Diffusion Model (RDM) for the second, long-lasting diffusion phase in pseudorapidity space [1–5] is reviewed in the next section.

#### 4 Linear diffusion phase

Since the initial power-law behaviour is superseded by the Gaussian evolution at sufficiently large times, the evolution is started here at  $t=0$  with  $\delta$ -function or gaussian initial conditions to illustrate the outcome of the three-sources model for large times and in particular, to compare to data.

The situation at moderate times with  $\delta$ -function initial conditions is shown in the lower part of Fig. 2, with separate Gaussians in the three sources in rapidity space which have the same particle-number content as the initial power-law solutions of the nonlinear particle-production problem. The subsequent time evolution of these three sources leads to agreement with the experimental data at the interaction time  $t = \tau_{\text{int}}$ , and to statistical equilibrium for  $t \rightarrow \infty$ .

The nonequilibrium-statistical description of this evolution is based on an essentially linear diffusion equation which is briefly reviewed in this section. We have used a Fokker-Planck equation (Uhlenbeck-Ornstein [29] version with  $\kappa = 0$ ) [1–5] for the distribution function  $R(y, t)$  for produced charged hadrons in rapidity space

$$\frac{\partial}{\partial t} R(y, t) = -\nabla_y [J(y)R(y, t)] + D_y \nabla_y^2 R(y, t). \quad (28)$$

The drift is now taken into account since we look at the large-time behaviour, and the drift function  $J(y)$  determines the speed of the statistical equilibration in  $y$ -space. In order to attain the Boltzmann distribution for large times, the drift term must have the form [8, 39]

$$J(y) \propto m_{\perp} \sinh(y) \propto p_{\parallel} \quad (29)$$

with the transverse mass  $m_{\perp} = \sqrt{(m^2 + p_{\perp}^2)}$ , and the longitudinal momentum  $p_{\parallel}$ . This introduces another nonlinearity into the problem, which prohibits an analytical solution of the diffusion equation. Such an analytical solution [1] is, however, possible by linearising the drift function in a relaxation ansatz

$$J(y) = (y_{\text{eq}} - y)/\tau_y \quad (30)$$

with the rapidity relaxation time  $\tau_y$ , and the equilibrium value of the rapidity  $y_{\text{eq}}$  that is calculated from energy and momentum conservation in the system of participants. The deviations of the solutions for nonlinear and linear versions of the drift are not pronounced [39] and hence, I have used the analytical solutions of the linear problem for the components  $R_k(y, t)$  ( $k = 1, 2, 3$ ) of the distribution function

$$\frac{\partial}{\partial t} R_k(y, t) = \frac{1}{\tau_y} \frac{\partial}{\partial y} \left[ (y - y_{\text{eq}}) \cdot R_k(y, t) \right] + D_y^k \frac{\partial^2}{\partial y^2} R_k(y, t). \quad (31)$$

The diagonal components  $D_y^k$  of the rapidity diffusion tensor contain the microscopic physics in the respective beam-like ( $k = 1, 2$ ) and central ( $k = 3$ ) regions. They account for the statistical broadening of the distribution functions. To connect with data, one has to consider the additional broadening due to longitudinal collective expansion that leads to a larger (effective) value of  $D_y$  [40] than what is calculated [1] from the dissipation-fluctuation theorem (Einstein relation).

As discussed above, the initial conditions in the linear phase are taken as  $R_{1,2}(y, t = 0) = \delta(y \pm y_{\text{max}})$  with the maximum rapidity  $y_{\text{max}} = 5.36$  at the highest RHIC energy of  $\sqrt{s_{NN}} = 200$  GeV (beam rapidities are  $y_{1,2} = \mp y_{\text{max}}$ ), and  $R_3(y, t = 0) = \delta(y)$ . A midrapidity gluon-dominated symmetric source had also been proposed by Bialas and Czyz [30].

This initial condition for the midrapidity source in the linear case corresponds to initial particle production without any longitudinal motion, independently of the mass of the collision partners: the third source is created at  $y = 0$ , and the drift towards the equilibrium value  $y = y_{\text{eq}}$ , as well as the rapid collective expansion, sets in subsequently. (In contrast, the nonlinear model as discussed in the previous section produces power-law tails at short times, which are superseded by the gaussian tails of the linear evolution only at later times).

The mean values in the three sources have the time dependence

$$\langle y_{1,2}(t) \rangle = y_{\text{eq}} [1 - \exp(-t/\tau_y)] \mp y_{\text{max}} \exp(-t/\tau_y) \quad (32)$$

for the sources (1) and (2), and

$$\langle y_3(t) \rangle = y_{\text{eq}} [1 - \exp(-t/\tau_y)] \quad (33)$$

for the moving central source. The three mean values reach the equilibrium limit for time to infinity. In our previous RDM-calculation [3] with slightly different initial condition, the mean value of the central source was at the equilibrium limit  $\langle y_3(t) \rangle = y_{\text{eq}}$  independently of time, thus assuming instant equilibration in this source regarding the mean values. The variances  $\sigma_{1,2,3}^2(t)$  are as in Eq.(24), with  $D_y \rightarrow D_y^p$ .

It turns out that for d+Au at the highest RHIC energy, one can not determine from a comparison with the data which of the two possibilities for the initial conditions of the central source is more realistic because the  $\chi^2$  is nearly identical in both cases. The subsequent diffusion-model time evolution in pseudorapidity space is followed up to the interaction time  $\tau_{\text{int}}$ , when the produced charged hadrons cease to interact strongly.

The quotient  $\tau_{\text{int}}/\tau_y$  is determined from the minimum  $\chi^2$  with respect to the data, simultaneously with the minimization of the other free parameters - namely, the variances of the three partial distribution functions, and the number of particles produced in the central source. In this nonequilibrium-statistical approach, the equilibrium value of the rapidity and its dependence on centrality is calculated from energy and momentum conservation in the system of participants as

$$y_{\text{eq}}(b) = \frac{1}{2} \ln \frac{\langle m_1^T(b) \rangle \exp(-y_{\text{max}}) + \langle m_2^T(b) \rangle \exp(y_{\text{max}})}{\langle m_2^T(b) \rangle \exp(-y_{\text{max}}) + \langle m_1^T(b) \rangle \exp(y_{\text{max}})} \quad (34)$$

with the transverse masses  $\langle m_{1,2}^T(b) \rangle = \sqrt{m_{1,2}^2(b) + \langle p_T \rangle^2}$ , and masses  $m_{1,2}(b)$  of the "target"(Au)- and "projectile"(d)-participants that depend on the impact parameter  $b$ . The average numbers of participants

**Table 1** Produced charged hadrons in minimum-bias d + Au collisions at  $\sqrt{s_{NN}} = 200$  GeV,  $y_{1,2} = \mp 5.36$  in the linear Relativistic Diffusion Model. The equilibrium value of the rapidity in the RDM is  $y_{eq}$ , the time parameter (see text) is  $p$ , the corresponding value of interaction time over relaxation time is  $\tau_{int}/\tau_y$ , the variance of the central source in  $y$ -space is  $\sigma_3^2$ . The number of produced charged particles is  $N_{ch}^{1,2}$  for the sources 1 and 2 and  $N_{ch}^3$  for the central source, the percentage of charged particles produced in the midrapidity source is  $n_{ch}^3$ , and  $\chi^2/d.o.f.$  is the result of the minimization [3] per number of degrees of freedom.

$y_{eq}$	$p$	$\tau_{int}/\tau_y$	$\sigma_3^2$	$N_{ch}^1$	$N_{ch}^2$	$N_{ch}^3$	$n_{ch}^3(\%)$	$\chi^2/d.o.f.$
-0.664	0.54	0.78	4.19	55	14	22	24	2.44/48

$\langle N_{1,2}(b) \rangle$  from the Glauber calculations reported in [31] for minimum bias d + Au at the highest RHIC energy are  $\langle N_1 \rangle = 6.6$ ,  $\langle N_2 \rangle = 1.7$ , which we had also used in [3].

The average numbers of charged particles in the target- and projectile-like regions  $N_{ch}^{1,2}$  are proportional to the respective numbers of participants  $N_{1,2}$ ,

$$N_{ch}^{1,2} = N_{1,2} \frac{(N_{ch}^{tot} - N_{ch}^{eq})}{(N_1 + N_2)} \quad (35)$$

with the constraint  $N_{ch}^{tot} = N_{ch}^1 + N_{ch}^2 + N_{ch}^{eq}$ . Here the total number of charged particles  $N_{ch}^{tot}$  is determined from the data. The average number of charged particles in the equilibrium source  $N_{ch}^{eq}$  is a free parameter that is optimized together with the variances and  $\tau_{int}/\tau_y$  in a  $\chi^2$ -fit of the data using the CERN minuit-code. The results are summarized in Table 1.

The FPE is solved analytically as outlined in [1], and the solutions are converted to pseudorapidity space. This conversion is required because particle identification is not available. The relation between scattering angle  $\theta$  and pseudorapidity  $\eta$  is  $\eta = -\ln[\tan(\theta/2)]$ . Here  $\theta$  is measured relative to the direction of the deuteron beam. Hence, particles that move in the direction of the gold beam have negative, particles that move in the deuteron direction have positive pseudorapidities. The conversion from  $y$ - to  $\eta$ - space of the rapidity density

$$\frac{dN}{d\eta} = \frac{p}{E} \frac{dN}{dy} = j(\eta, \langle m \rangle / \langle p_T \rangle) \frac{dN}{dy} \quad (36)$$

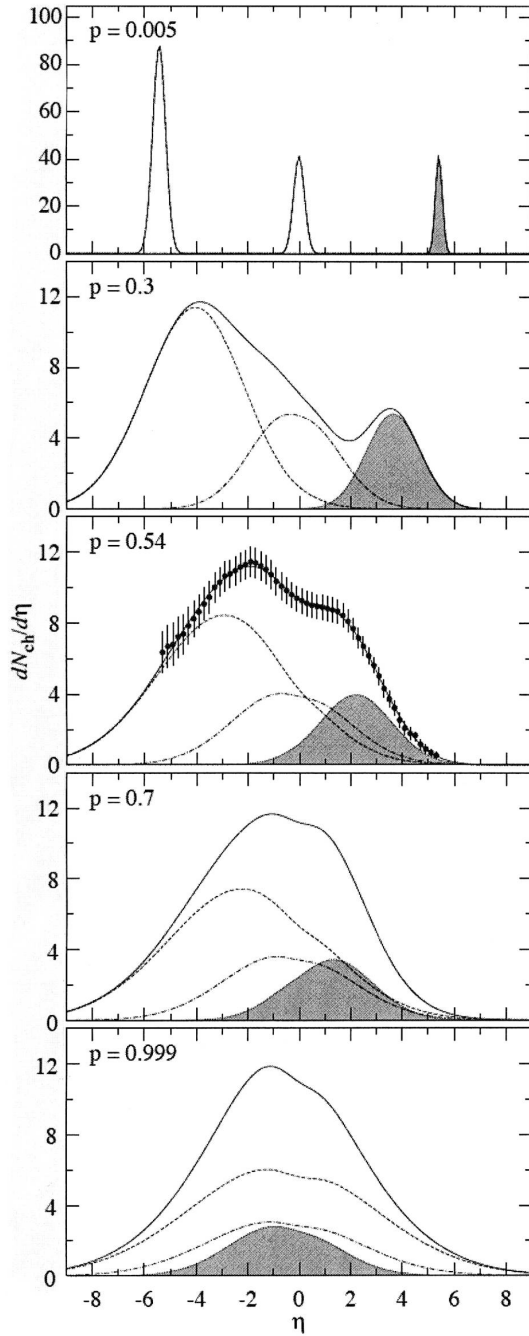
is performed through the Jacobian

$$j(\eta, \langle m \rangle / \langle p_T \rangle) = \cosh(\eta) \cdot [1 + (\langle m \rangle / \langle p_T \rangle)^2 + \sinh^2(\eta)]^{-1/2}. \quad (37)$$

The average mass  $\langle m \rangle$  of produced charged hadrons in the central region is approximated by the pion mass  $m_\pi$ , and a mean transverse momentum  $\langle p_T \rangle = 0.4$  GeV/c is used [3]. Due to the conversion, the partial distribution functions are different from Gaussians. The charged-particle distribution in rapidity space is obtained as incoherent superposition of nonequilibrium and local equilibrium solutions of (31)

$$\frac{dN_{ch}(y, t = \tau_{int})}{dy} = N_{ch}^1 R_1(y, \tau_{int}) + N_{ch}^2 R_2(y, \tau_{int}) + N_{ch}^3 R_3(y, \tau_{int}) \quad (38)$$

with the interaction time  $\tau_{int}$  (total integration time of the differential equation). The integration is stopped at the value of  $\tau_{int}/\tau_y$  that produces the minimum  $\chi^2$  with respect to the data and hence, the explicit value of  $\tau_{int}$  is not needed as an input. The resulting values for  $\tau_{int}/\tau_y$  are given in Table 1 together with the widths of the central distributions, and the particle numbers in the three sources.



**Fig. 3** Time evolution of pseudorapidity distributions for produced charged particles from minimum-bias d + Au collisions at  $\sqrt{s_{NN}} = 200$  GeV in the linear diffusion model as outlined in [3, 4]. The d-like source is shaded to illustrate the movement in  $\eta$ -space towards equilibrium. Dash-dotted curves show the slightly moving, gluon-dominated midrapidity source for hadron production. Results for five time steps ( $p$ -values, cf. text) are displayed. Agreement with the data [31] is reached at  $p = 0.54$ . Statistical equilibrium centered at  $\eta_{eq}$  would be achieved at later times. Reviewed from [4].

The time evolution is shown together with the comparison to PHOBOS minimum-bias data [31] in Fig. 3. It is evident that the two beam-like distribution functions move towards smaller absolute pseudorapidities as time increases, and reach agreement with the data at  $p = 0.54$ . Here the time evolution parameter

$p$  in the numerical calculation is defined as <sup>1</sup>

$$p = 1 - \exp(-t/\tau_y). \quad (39)$$

The minimum-bias result also shows the asymmetric shape of the distribution function, which is very well reproduced in the diffusion calculation. At larger values of the time evolution parameter  $p$ , all three subdistributions tend to become symmetric in  $y$  with respect to the equilibrium value  $y_{\text{eq}}$ , indicating the approach to thermal equilibrium. At  $p = 0.999$ , the equilibrium state is already closely approached. The slight asymmetry is due to the conversion from rapidity- to pseudorapidity space which tends to produce a dip at  $\eta = 0$ . For time to infinity, statistical equilibrium in pseudorapidity space would be reached.

We have shown in [3] that with this linear RDM approach, the centrality dependence of the measured pseudorapidity distributions [31] from central to very peripheral collisions can also be modeled in considerable detail, Fig. 4. For peripheral collisions, the asymmetry of the overall distribution is not yet pronounced because here the d- and the Au-like partial distributions are similar in size due to the small number of participants.

Towards more central collisions, the number of gold participants rises, and the corresponding partial distribution of produced particles becomes more important. In addition, the distributions drift towards the equilibrium value. Both effects produce the asymmetric shape, which is also seen in minimum-bias. The tails of the distribution functions are of gaussian shape in perfect agreement with the data. This shows that the power-law tails of the initial phase have not survived the time evolution, as is confirmed when the result of the initial phase is used explicitly as an initial condition for the linear evolution.

It is interesting to compare the behaviour of the rapidity or pseudorapidity distribution functions with results from different approaches to the problem such as saturation models [32, 33, 35], viscous hydrodynamics [37], or ideal hydrodynamics [38].

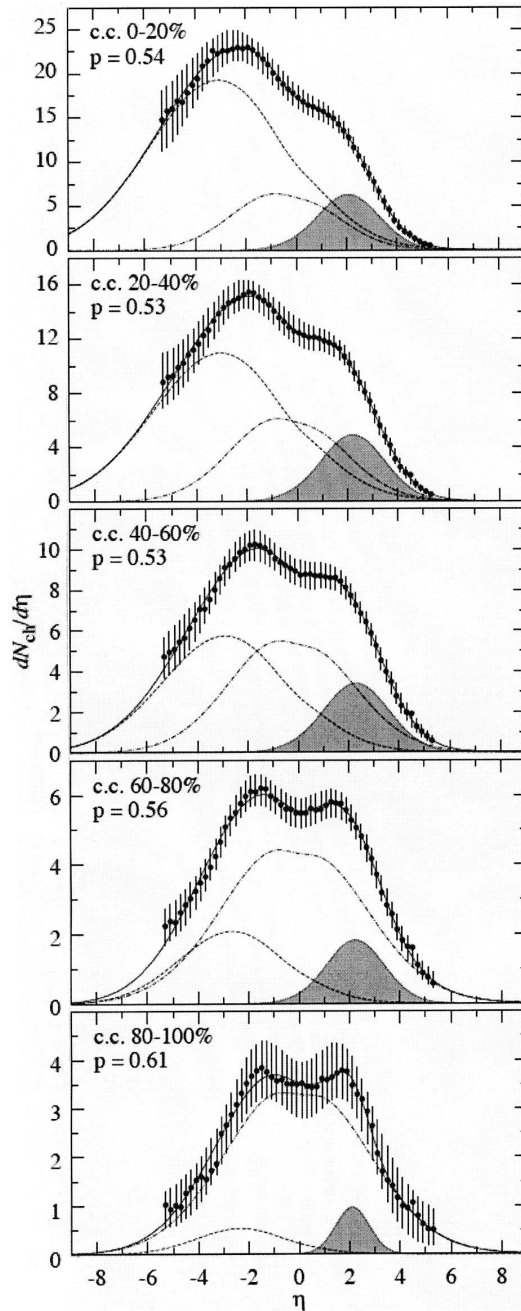
Calculations within the framework of the Parton Saturation Model not only predict the midrapidity value, but also the full rapidity distribution function (at RHIC energies, and also at LHC) [33, 34]. These calculations are based on a classical effective theory that describes the gluon distribution in large nuclei at high energies where saturation might occur at a critical momentum scale, to form a Color Glass Condensate (CGC) [35].

This assumption has a clear and reasonable physical basis and yields good results for pseudorapidity distribution functions of produced charged hadrons at the available RHIC energies. Problems may be expected for net-proton rapidity distributions since protons and antiprotons are produced in equal amounts from the CGC. At LHC energies, the overall pseudorapidity distribution from the CGC as obtained with the assumption of a constant  $\alpha_s$  for strong coupling is slightly narrower than the corresponding diffusion-model prediction [5].

Additional consideration of a running coupling gives a midrapidity value that is of the order of 10% smaller; another uncertainty arises from the extrapolation of the saturation scale to LHC energies. Various predictions for central rapidity densities and pseudorapidity distributions at RHIC and LHC energies had been summarized e.g. in [36], where also the differences among the existing models - including hydrodynamical and pQCD approaches and their numerical implementations - had been discussed.

In relation to these approaches, the analytical diffusion model provides good results when compared in detail to the experimental distribution functions at RHIC energies, in particular, in the tails of the distributions. To provide a microscopic foundation, however, a derivation of the diffusion coefficients in the three sources would be required. Due to the valence-quark dominance in the beam-like sources as opposed to the mainly gluonic midrapidity source, the diffusion coefficients may turn out to be substantially different in the three sources.

<sup>1</sup> There is a difference of a factor of two in the exponent as compared to the definition of  $p$  used in [3], which causes different  $t/\tau_y$  values for given  $p$ .



**Fig. 4** Calculated pseudorapidity distributions of produced charged particles in d + Au collisions at  $\sqrt{s_{NN}} = 200$  GeV for five different centralities, as outlined in [3, 4]. Central collisions are shown in the top frame, peripheral at the bottom. The linear diffusion-model (RDM) results for three sources (d-like source shaded) in  $\eta$ -space are shown together with their incoherent sums as  $\chi^2$ -minimizations at each centrality cut (c.c.). The time variable is  $p$  (see text). The initial conditions for the central source are slightly different from [3], see text. Data are from PHOBOS [31]. Reviewed from [4].

## 5 Conclusion

I have presented a nonlinear Relativistic Diffusion Model that includes an explicit analytical treatment of the initial parton-production phase in rapidity space. As a consequence of the high rapidity density at short times  $t \simeq 0.25$  fm/c, the rapidity diffusion coefficient depends on the distribution function, such that the problem is highly nonlinear in the initial phase.

For a power-law dependence of the diffusion coefficient on the distribution function with an exponent  $\kappa$ , the mathematical technique of similarity solutions that has been refined in many previous works proves to be useful in the present physical context. In particular, I have investigated the so-called source solution in rapidity space.

An adequate solution to the nonlinear problem should not have a free boundary in rapidity space that moves with finite velocity as is the case for positive values of  $\kappa$ , but in parton production there should be an instantaneous spread in  $y$ -space without a free boundary. This corresponds to the case  $\kappa = -1$ , which can be solved analytically using the technique of similarity solutions.

During particle production in three sources, the norm of the distribution function increases, which I have considered phenomenologically by letting the integration constant depend on particle number. Since particle production is very rapid - exponentially in time -, this increase of the norm of the distribution function turns out to be faster than the spread of the distribution function in rapidity space due to nonlinear diffusion and hence, the power-law tails of the distribution function remain small during the parton-production phase.

The result of the parton-production phase is then used as initial condition for the later (pre-hadronic and hadronic) stage of the collision, which is treated here in the linear relativistic diffusion model. The linear evolution washes out the initial power-law tails of the distribution function, which become gaussian.

With the proper Jacobi transformation to pseudorapidity space, this approach yields very precise agreement with charged-hadron data for both asymmetric systems (d+Au), and symmetric systems such as Cu+Cu and Au+Au [2, 5]. It is also particularly suitable for predictions at LHC energies of 5.5 TeV for Pb+Pb.

The second collision phase lasts about 7-10 fm/c depending on the system, the incident energy, and the centrality. Due to the schematic treatment that is based on a linear partial differential equation, particle number is conserved in this phase, which appears as a reasonable physical assumption even though it is certainly not strictly valid.

For time to infinity, the evolution of the distribution function proceeds to statistical equilibrium with respect to the variable rapidity or pseudorapidity. Comparing the data with this time evolution as modeled by the solutions of the linear problem, it is evident that at the time of the measurement - when strong interaction ceases - the system is still far from statistical equilibrium. This underlines the view that relativistic heavy-ion collisions are very suitable to observe strongly interacting many-body systems with a large amount of particle production on their way to statistical equilibrium.

In this work I have not considered the connection between the diffusion approach and QCD. It is obvious that the forward and backward sources for produced particles are related to the valence quarks, whereas the central source is essentially due to gluon-gluon collisions. An actual microscopic calculation of the three sources emphasizing their relative size (number of produced particles) is therefore of interest [41].

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